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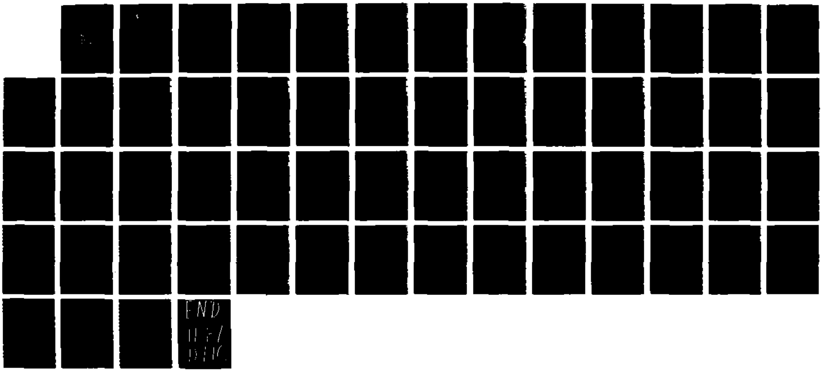

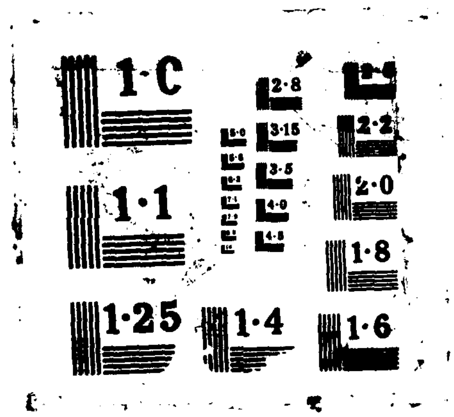
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The structure of an incompressible turbulent boundary layer subjected to a strong adverse pressure gradient is studied by means of an asymptotic analysis of the Reynolds time-averaged equations. Limit-process expansions developed in the limit of large Reynolds number reveal a relatively thick nondefect layer in the outer region of the boundary layer near the exterior inviscid flow, and a relatively thin layer near the wall. To leading orders of approximation, the momentum balance involves convection, pressure gradient, and turbulent stress in the outer layer, and pressure gradient, and turbulent and viscous stresses in

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19. the inner layer. The asymptotic expansions for these two layers are matched in an arbitrary intermediate region, wherein the streamwise velocity has a square-root dependence and the Reynolds stress has a corresponding linear dependence on the normal coordinate. The leading-order approximations for the outer and inner layers give rise to similarity formulations, from which appropriate similarity formulations for the distinguished intermediate layer have been identified and developed. These latter formulations are employed to analyze available experimental data.

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ASYMPTOTIC ANALYSIS OF A TURBULENT BOUNDARY LAYER IN A STRONG ADVERSE PRESSURE GRADIENT

W. B. Bush and L. Krishnamurthy

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UNIVERSITY OF DAYTON
RESEARCH INSTITUTE
DAYTON, OHIO 45469-0001

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PREFACE

The completion of this research and the documentation of the results were supported by the Research Funds of the Office of Director, University of Dayton Research Institute. Dr. William B. Bush, Senior Associate, King, Buck & Associates, Inc., 2384 San Diego Ave., Suite 2, San Diego, CA, 92110, and Dr. L. Krishnamurthy, Senior Research Engineer, Applied Physics Division, University of Dayton Research Institute express their appreciation to Drs. John Wurst and Eugene Gerber for their encouragement. One of us (L. K.) was supported in part by the U. S. Air Force Office of Scientific Research under Contract No. F-49620-85-C-0137 (Major John Thomas, Jr., Project Monitor). The authors are indebted to Ms. Ellen Bordewisch for her skillful assistance in the preparation of this report. The authors are pleased to note that the present analysis and results have benefited from earlier interactions with Drs. Noor Afzal and Francis E. Fendell.

1. INTRODUCTION

The vast literature that addresses turbulent boundary layers developing under significant longitudinal (favorable or adverse) pressure gradients attests to their (practical and fundamental) importance. Much is known about these flows from the experimental, theoretical, and computational studies of the past 30 years, yet no satisfactory theory exists for turbulent boundary layers with strong pressure gradients. The fundamental difficulty, to be sure, stems from the general closure problem for the time-averaged equations. The recog ition of this difficulty has led to the development of semiempirical model equations of the turbulence and computation thereof, especially since the publication of the two-volume proceedings of the 1968 AFOSR-IFP Stanford Conference (Kline et al. 1969; Coles & Hirst 1969) and the increasing availability of larger and faster computing resources. This effort apart, the theoretical approaches resting on firmer grounds for addressing pressure-gradient boundary layers have been twofold. The first is based upon similarity analyses and dimensional arguments (see, e.g., Yaglom 1979). The second is based upon a systematic treatment by the method of matched asymptotic expansions (see, e.g., Van Dyke 1975; Kevorkian & Cole 1981). The present paper is concerned with exploiting this latter approach, in the limit of large (turbulent) Reynolds number, to study a turbulent boundary layer under a strong adverse pressure gradient in a steady, two-dimensional, incompressible flow past a smooth, nonporous wall.

The general adverse-pressure-gradient problem, in equilibrium or self-similar form, initially conjectured by Clauser (1954), from an analogy with the zero-pressure-gradient case, has been analyzed in detail by Townsend (1956a,b, 1960, 1961a,b) and Rotta (1962). Stratford (1959a,b) first studied, both theoretically and experimentally, boundary layers held at incipient separation, the limiting or strong adverse-pressure-gradient case for equilibrium layers. Further discussions of the theoretical and experimental results and their implications for this limiting-case problem are summarized in Clauser (1956), Mellor & Gibson (1966), Mellor (1966), Townsend (1976), Kader & Yaglom (1978), Schofield (1981), and Afzal (1983), among others. Indeed, the results of this paper are compared with those of Kader & Yaglom, obtained from similarity and

dimensional arguments, and with those of Afzal, obtained from asymptotic analysis, as well as with the original results of Stratford.

The asymptotic analysis herein for the "sharply rising pressure" case of Stratford is based upon the Reynolds time-averaged equations for steady two-dimensional flow. For this case, as seen in figure 1, in addition to the exterior (or inviscid-flow) region, there is the interior (or turbulent boundary-layer-flow) region that consists of two principal layers. The first principal layer is the relatively thick nondefect outer layer, characterized, to leading order of approximation, by a convection--pressure-gradient--turbulent-stress momentum balance. The second principal layer is the relatively thin inner layer, in which, to leading order of approximation, the pressure gradient and the turbulent and viscous stresses constitute the momentum balance.

In Section 2, the steady, two-dimensional equations of continuity and mean transfer of x- and y- momentum and the complementary boundary conditions for the problem are presented (see Yajnik 1970; Mellor 1972). In Sections 3 and 4, the asymptotic analyses for the flow in the exterior inviscid and interior viscous (or boundary-layer) regions are presented, once the orderings of the pertinent parameters are identified.

For the boundary-layer region, the postulated characterizations of the flow variables in the outer and inner layers are presented in Sections 4.1 and 4.2, respectively. When it is taken that the streamwise velocity has a square-root variation with respect to the normal coordinate in an arbitrary "overlap" region intermediate to the outer and inner layers, and that the Reynolds stress has a corresponding linear variation, as is observed experimentally (see, e.g., Stratford 1959a,b), the matching of the postulated zeroth-order outer-layer and inner-layer solutions for these flow quantitites is demonstrated (see Section 4.3). Indeed, this matching provides the inner-edge boundary conditions for the higher-order outer-layer boundary-value problems <u>and</u> the outer-edge boundary conditions for the higher-order inner-layer boundary-value problems. In the analysis of Section 4.1, a model of the turbulent length-scale function, consistent with the experimentally determined nearfield behavior of the outer-layer flow field and with the mixing-length hypothesis (for the eddy

diffusivity), is developed, as is a model for the eddy viscosity in the farfield of the outer layer (see, e.g., Clauser 1956; Mellor & Gibson 1966; Mellor 1966).

In Sections 5.1 and 5.2, respectively, similarity formulations for the zeroth-order approximations of the asymptotic representations for the outer and inner layers are presented. For previous work on such zeroth-order similarity formulations for both the outer and the inner layers, see, e.g., Townsend (1976), Mellor & Gibson (1966), Mellor (1966), and Afzal (1983).

For the zeroth-order approximation for the outer layer, subject to the introduction of the aforementioned model closure, which incorporates the experimentally observed nearfield and farfield behaviors of the streamwise velocity and Reynolds stress, the similarity boundary-value problem becomes an eigenvalue problem for the determination of H_0, the (constant) leading-order shape factor, where this shape factor, H_0, is related to the leading-order streamwise-pressure distribution, $p_0^*(x)$, by

$$H_0 = [\{1-2p_0^*(x)\}\{p_0^{*\prime}(x)\}^{-2}\{-p_0^{*\prime\prime}(x)\}-4] = \text{const.}$$

Thus, the similarity formulation holds for only a specific pressure distribution, $p_0^*(x)$, and/or outer-edge streamwise-velocity distribution, $u_0^*(x)$, i.e.,

$$p_0^*(x) = \frac{1}{2}[1 - u_{00}^{*2}(x-x_0)^{-2/(H_0+2)}] \text{ and/or}$$

$$u_0^*(x) = \{1 - 2p_0^*(x)\}^{1/2} = u_{00}^*(x - x_0)^{-1/(H_0+2)}.$$

This velocity relation was first obtained by Townsend (1960). The normal-length function, $h_0^*(x)$, is (classically) linear in x, i.e.,

$$h_0^*(x) = [\{1 - 2p_0^*(x)\}\{p_0^{*\prime}(x)\}^{-1}] = (H_0 + 2)(x - x_0).$$

As to the derivatives of this pressure distribution, similarity requires

$$p_0^{*\prime}(x) = \frac{u_{00}^{*2}}{(H_0 + 2)} (x - x_0)^{-(H_0+4)/(H_0+2)} > 0,$$

$$p_0^{*\prime\prime}(x) = - \frac{(H_0 + 4)u_{00}^{*2}}{(H_0 + 2)^2} (x - x_0)^{-2(H_0+3)/(H_0+2)} < 0, \ \ldots .$$

With $p_0^{*\prime\prime}(x) < 0$, the present similarity formulation holds for a decreasing adverse pressure gradient, rather than an increasing one, as studied by Samuel & Joubert (1974). Further, $p_0^{*\prime\prime}(x) < 0$ violates the moving-equilibrium assumption (Kader & Yaglom 1978; Yaglom 1979), wherein it is taken that $p_0^{*\prime}(x)$ varies only slowly with x.

The inner-layer zeroth-order-approximation similarity formulation follows without the adoption of a closure hypothesis, in that only normal derivatives of unknown functions appear explicitly in the zeroth-order problem of Section 4.2. However, this similarity formulation is based upon a square-root variation of the streamwise velocity (with respect to the appropriately scaled normal coordinate) and a linear one of the Reynolds stress at the outer edge of this inner layer. These variations are consistent with the asymptotic behavior of an inner-layer mixing-length-hypothesis closure. To display the detailed farfield and nearfield behaviors of the flow quantities, here, an inner-layer dissipation-factor closure model (see, e.g., Van Driest 1956; Patankar & Spalding 1970; Szablewski 1970) is adopted. From the present formulation, the characteristic streamwise-velocity and normal-length functions for this inner layer are, respectively,

$$u_{p0}^{*}(x) \propto \{p_0^{*\prime}(x)\}^{1/3} \propto (x - x_0)^{-(H_0+4)/3(H_0+2)} ,$$

$$h_{p0}^{*}(x) \propto \{p^{*\prime}(x)\}^{-1/3} \propto (x - x_0)^{(H_0+4)/3(H_0+2)} .$$

Since the matching of the zeroth-order outer- and inner-layer solutions is established (in Section 4.3) without the requirement of similarity, the matching

also holds upon the introduction of similarity. In the analyses of Townsend (1976), Mellor & Gibson (1966), and Mellor (1966), the corresponding outer- and inner-layer similarity solutions are patched (at a point of assumed common validity), rather than matched. In the analysis of Afzal (1983), the zeroth-order inner-layer similarity boundary-value problem is essentially correct; however, because the matching requirement for the streamwise velocity is not correctly formulated, the given complementary outer-layer similarity boundary-value problem is subject to incorrect inner-edge boundary conditions. Afzal matches the (appropriately scaled) normal derivative of the inner-edge limit of the outer-layer streamwise-velocity solution and that of the outer-edge limit of the inner-layer streamwise-velocity solution. The correct procedure (see, e.g., Kevorkian & Cole 1981) requires the direct matching of the inner-edge limit of the outer-layer streamwise-velocity solution and the outer-edge limit of the inner-layer streamwise-velocity solution in an arbitrary intermediate layer. The same procedure is required for the matching of the corresponding Reynolds-stress solutions. In the present case, the introduction of Stratford's empirical results and/or the specification of the mixing-length-hypothesis closure provides the limiting behaviors of the outer- and inner-layer solutions, and the zeroth-order matching is accomplished. In turn, it is possible to determine the appropriate higher-order terms in the outer- and inner-layer asymptotic expansions. These expansions are introduced in Section 4; their correctness is confirmed in Section 6.

From the analyses of Sections 4.1 and 4.2, the equations of motion for the higher-order approximations for both the outer and the inner layers of the interior viscous region are obtained. From matching with the exterior-inviscid-region solutions, the boundary conditions at the outer edge of the outer layer are obtained; the boundary conditions at the inner edge of the inner layer (i.e., the wall) follow directly from the original boundary-value problem; and the higher-order-approximation boundary conditions at the inner edge of the outer layer follow from matching with the outer-edge solutions of the zeroth-order approximation for the inner layer, just as the higher-order-approximation boundary conditions at the outer edge of the inner layer follow from matching with the inner-edge solutions of the zeroth-order approximation for the outer layer. With this information and that obtained from the zeroth-order outer- and

inner-layer similarity boundary-value problems, it is possible to develop a sequence of higher-order similarity boundary-value problems for both the outer and the inner layers. The details of this development are presented in Section 6. It is shown that the outer-layer solutions for the streamwise-velocity and Reynolds-stress functions are of the following forms:

$$\frac{u}{u_0^*} \cong \frac{df_0}{d\eta} + \Lambda_*^{1/2}\frac{df_1}{d\eta} + \Lambda_*\frac{df_2}{d\eta} + \Lambda_*^{3/2}\frac{df_3}{d\eta} + \dots ,$$

$$\frac{\tau}{u_0^{*2}} \cong \phi_0 + \Lambda_*^{1/2}\phi_1 + \Lambda_*\phi_2 + \Lambda_*^{3/2}\phi_3 + \dots .$$

Here, $u = u(x,y)$ and $\tau = \tau(x,y)$ are the (nondimensional) streamwise velocity and turbulent shear stress in the outer layer, respectively; $u_0^* = u_0^*(x) = \{1 - 2p_0^*(x)\}^{1/2}$ is the zeroth-order outer-edge streamwise velocity; $\eta = y/h_0^*$ is the outer-layer similarity coordinate, where y is the appropriately scaled outer-layer normal coordinate and $h_0^* = h_0^*(x) = [\{1 - 2p_0^*(x)\}\{p_0^{*'}(x)\}^{-1}]$ is the normal-length function for this layer; $f_k = f_k(\eta)$ and $\phi_k = \phi_k(\eta)$ are kth-approximation similarity streamfunction and Reynolds stress, respectively; and

$$\Lambda_* = \Lambda_*(x;\delta) = \delta[\{1 - 2p_0^*(x)\}^{-1}\{p_0^{*'}(x)\}^{2/3}]$$

$$= \delta[(H_0 + 2)\, u_{00}^*(x - x_0)^{(H_0+1)/(H_0+2)}]^{-2/3},$$

where $\delta = (\tilde{\delta}/\tilde{c}) = (\tilde{u}_\infty\tilde{c}/\tilde{\nu})^{-2/7}$ is the nondimensional thickness of this outer layer. It is also shown that the corresponding inner-layer solutions for the streamwise-velocity and Reynolds-stress functions are of the following forms:

$$\frac{u}{u_{p0}^*} \cong \frac{dg_0}{d\zeta} + \Lambda_*\frac{dg_2}{d\zeta} + \dots ,$$

$$\frac{\tau}{u_{p0}^{*2}} \cong \omega_0 + \Lambda_*\,\omega_2 + \dots .$$

Here, $u = u(x,r)$ and $\tau = \tau(x,r)$ are the (nondimensional) streamwise velocity and turbulent shear stress in the inner layer, respectively; $u^*_{p0} = u^*_{p0}(x;\delta) = \delta^{1/2}\{p_0^{*\prime}(x)\}^{1/3}$ is the (so-called) pressure velocity, the characteristic zeroth-order streamwise velocity of the inner layer; $\zeta = r/h^*_{p0}$ is the inner-layer similarity coordinate, where $r = y/\delta$ is the inner-layer normal coordinate and $h^*_{p0} = h^*_{p0}(x) = \{p_0^{*\prime}(x)\}^{-1/3}$ is the normal-length function for this layer; and $g_k = g_k(\zeta)$ and $\omega_k = \omega_k(\zeta)$ are the kth-approximation similarity streamfunction and Reynolds stress, respectively.

Two important results are obtained from the similarity formulations of Section 6: (1) the similarity coordinates, η and ζ, respectively, are related, through the expansion parameter, Λ_*, by

$$\eta = \Lambda_*\zeta \text{ and/or } \zeta = \Lambda_*^{-1}\eta;$$

and (2), the characteristic streamwise velocitites, u^*_0 and u^*_{p0}, are also related, through the expansion parameter, Λ_*, by

$$u^*_{p0} = \Lambda_*^{1/2}u^*_0 \text{ and/or } u^*_0 = \Lambda_*^{-1/2}u^*_{p0} .$$

In Section 7, the solutions for the streamwise velocity and the Reynolds stress are presented for the distinguished similarity intermediate layer,† the existence of which is suggested by the inner-edge behaviors of the outer-layer expansions and/or the outer-edge behaviors of the inner-layer expansions, found in Section 6. For this intermediate layer, the similarity normal coordinate is

$$\chi = (\eta\zeta)^{1/2} = \Lambda_*^{-1/2}\eta = \Lambda_*^{1/2}\zeta,$$

† For a discussion of three-layer, or intermediate-layer, theory, within the framework of (classical) two-layer theory for turbulent pipe/channel and favorable-pressure-gradient boundary-layer flows, see, e.g., Afzal (1982); Afzal & Bush (1985).

and the appropriate speed function is

$$u_{10}^* = (u_0^* u_{p0}^*)^{1/2} = \Lambda_*^{1/4} u_0^* = \Lambda_*^{-1/4} u_{p0}^*.$$

In turn, the intermediate-layer solutions for the streamwise-velocity and the Reynolds-stress functions are

$$\frac{u}{u_{10}^*} \cong [\beta_{01}^+ \chi^{1/2}] + \Lambda_*^{1/4}[\beta_{10}^+]$$

$$+ \Lambda_*^{1/2}[\beta_{03}^+ \chi^{3/2} + \alpha_{21}^+ \chi^{-1/2}] + \Lambda_*^{3/4}[\beta_{12}^+ \chi + \alpha_{32}^+ \chi^{-1}] + \dots ,$$

$$\frac{\tau}{u_{10}^{*2}} \cong [\phi_{02}^+ \chi] + \Lambda_*^{1/2}[\phi_{04}^+ \chi^2 + \phi_{20}^+] + \Lambda_*^{3/4}[\phi_{13}^+ \chi^{3/2} + \lambda_{31}^+ \chi^{-1/2}] + \dots ,$$

where β_{jk}^+, α_{jk}^+, and ϕ_{jk}^+, λ_{jk}^+ are consts., whose values depend upon the outer- and inner-layer closures employed. Consistent with Stratford's data and the mixing-length-hypothesis closure,

$$\beta_{01}^+ = 2/\kappa \ (\approx 4.9 \text{ for } \kappa \approx 0.41).$$

The representations of Kader & Yaglom (1978), Afzal (1983), and Stratford (1959b) for the experimental data on the streamwise velocity in the intermediate layer are re-cast in the form of the aforementioned intermediate-layer streamwise-velocity solutions. The results indicate that predictions based on the three-layer theory are better than those based on the (classical) two-layer theory.

2. EQUATIONS OF MEAN MOTION

Consider the steady two-dimensional fully developed turbulent flow of an incompressible fluid of constant density and (kinematic) viscosity ($\tilde{\rho}$, $\tilde{\nu}$ = consts.) past a smooth flat surface (cf. Yajnik 1970; Mellor 1972). Let $\tilde{X} = \tilde{c}\,X$ and $\tilde{Y} = \tilde{c}\,Y$ represent the coordinates tangential and normal to the surface, respectively, with $\tilde{c}$ a characteristic body length. The mean velocity components in the $\tilde{X}$- and $\tilde{Y}$-directions, respectively, and the mean pressure are $\tilde{U} = \tilde{u}_\infty U$, $\tilde{V} = \tilde{u}_\infty V$, $\tilde{P} = \tilde{p}_\infty + \tilde{\rho}\tilde{u}_\infty^2 P$, with $\tilde{u}_\infty$ and $\tilde{p}_\infty$ being the velocity and pressure in the undisturbed region far from the surface. The laminar stress tensor components are $\tilde{S}_{ij} = \tilde{\rho}\tilde{u}_\infty^2(\tilde{u}_\ell/\tilde{u}_\infty)S_{ij}$ (i or j = $\tilde{X}$, $\tilde{Y}$ and/or X, Y), with $S_{ij} = \partial U_i/\partial X_j$ and $\tilde{u}_\ell = \tilde{\nu}/\tilde{c}$, a characteristic "laminar velocity"; the turbulent stress tensor components are $\tilde{T}_{ij} = \tilde{\rho}\tilde{u}_\infty^2(\tilde{u}_t/\tilde{u}_\infty)^2 T_{ij}$ (i or j = $\tilde{X}$, $\tilde{Y}$ and/or X, Y), with $\tilde{u}_t$ a characteristic "turbulent velocity." In the analysis that follows, it is taken that the turbulent and laminar velocity parameters, $\varepsilon = (\tilde{u}_t/\tilde{u}_\infty)$ and $\mu = (\tilde{u}_\ell/\tilde{u}_\infty) = (\tilde{\nu}/\tilde{u}_\infty\tilde{c}) = R^{-1}$, respectively, go to zero.

In the domain $X > X_0$, $0 < Y < \infty$, the (nondimensional) equations of mean motion for this flow are:

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0: \quad U = \frac{\partial \Psi}{\partial Y},\ V = -\frac{\partial \Psi}{\partial X} \ ; \tag{2.1}$$

$$\left(U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y}\right) + \frac{\partial P}{\partial X} = \varepsilon^2 \left(\frac{\partial T_{XX}}{\partial X} + \frac{\partial T_{XY}}{\partial Y}\right) + \mu \left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2}\right) , \tag{2.2a}$$

$$\left(U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y}\right) + \frac{\partial P}{\partial Y} = \varepsilon^2 \left(\frac{\partial T_{YX}}{\partial X} + \frac{\partial T_{YY}}{\partial Y}\right) + \mu \left(\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2}\right) . \tag{2.2b}$$

The surface boundary conditions are:

$$U,\ V \to 0,\ T_{ij} \to 0 \quad \text{as } Y \to 0 \ (X > X_0); \tag{2.3a}$$

the freestream boundary conditions are:

$$U \to 1,\ V \to 0,\ P \to 0,\ T_{ij} \to 0 \ \text{as } Y \to \infty \ (X > X_0). \tag{2.3b}$$

The upstream boundary conditions are taken to be

$$U,\ V,\ P,\ T_{ij} \rightarrow \text{fncs}(Y) \quad \text{as } X \rightarrow X_0 \ (0 < Y < \infty). \tag{2.3c}$$

In the present analysis, the existence of an exterior (or inviscid-flow) region and an interior (or turbulent boundary-layer-flow) region is stipulated. Further, it is taken that this boundary-layer region consists of two principal layers. By means of limit-process-expansion techniques, the solutions for the flow quantities in the exterior region are studied for

$$X,\ Y \text{ fixed as } \varepsilon,\ \mu, \text{ and } \delta(\varepsilon,\mu) \rightarrow 0; \tag{2.4}$$

those in the two principal layers of the interior region, respectively, are studied for

$$x,\ y \text{ fixed, with } x = X,\ y = Y/\delta, \text{ as } \varepsilon,\ \mu, \text{ and } \delta(\varepsilon,\mu) \rightarrow 0; \tag{2.5a}$$

$$x,\ r \text{ fixed, with } x = X,\ r = Y/\delta^2, \text{ as } \varepsilon,\ \mu, \text{ and } \delta(\varepsilon,\mu) \rightarrow 0. \tag{2.5b}$$

Here, $\delta(\varepsilon,\mu)$ is the effective-thickness parameter of the interior region, as well as that of the outer layer of this region; $\delta^2(\varepsilon,\mu)$ is the effective-thickness parameter of the inner layer of this interior region.

In this study, it is determined that ε and μ are not independent parameters; rather, it is determined that $\varepsilon = \varepsilon(\mu)$, and, in turn, $\delta(\varepsilon,\mu) = \delta(\mu)$. The following analysis requires that

$$\varepsilon(\delta) = \delta^{1/2}, \text{ and } \mu(\delta) = \delta^{7/2}, \tag{2.6a}$$

that is,

$$\varepsilon(\mu) = \mu^{1/7}, \text{ and } \delta(\mu) = \mu^{2/7}. \tag{2.6b}$$

3. THE EXTERIOR REGION

For X, Y fixed, with ε, $\mu \to 0$, the exterior-region expansions for the mean-velocity components, U, V, and the pressure, P, are in integral powers of the boundary-layer-thickness parameter, $\delta = \delta(\varepsilon,\mu)$, namely:

$$G(X, Y; \varepsilon, \mu) = G(X, Y; \delta) \cong G_0(X, Y) + \delta G_1(X, Y) + \delta^2 G_2(X, Y) + \dots , \tag{3.1}$$

where G = U, V, P. With the stipulation that there is no exterior-region turbulence and that the interior-region turbulence decays exponentially as the exterior region is approached, it is taken that the exterior-region expansions for the turbulent-stress components are in transcendentally small powers of δ, such that

$$T_{ij}(X, Y; \varepsilon, \mu) = T_{ij}(X, Y; \delta) \sim 0. \tag{3.2}$$

Substitution of the expansions of (3.1) and (3.2) into (2.1) and (2.2) yields equations for the successive exterior-region approximations. The equations for the *zeroth-*, *first-*, and *second-order* approximations, respectively, are

$$\frac{\partial U_0}{\partial X} + \frac{\partial V_0}{\partial Y} = 0: \quad U = \frac{\partial \Psi_0}{\partial Y}, \; V_0 = -\frac{\partial \Psi_0}{\partial X}, \tag{3.3a}$$

$$(U_0 \frac{\partial U_0}{\partial X} + V_0 \frac{\partial U_0}{\partial Y}) + \frac{\partial P_0}{\partial X} = 0, \tag{3.3b}$$

$$(U_0 \frac{\partial V_0}{\partial X} + V_0 \frac{\partial V_0}{\partial Y}) + \frac{\partial P_0}{\partial Y} = 0; \tag{3.3c}$$

$$\frac{\partial U_1}{\partial X} + \frac{\partial V_1}{\partial Y} = 0: \quad U_1 = \frac{\partial \Psi_1}{\partial Y}, \; V_1 = -\frac{\partial \Psi_1}{\partial X}, \tag{3.4a}$$

$$(U_0 \frac{\partial U_1}{\partial X} + V_0 \frac{\partial U_1}{\partial Y} + \frac{\partial U_0}{\partial X} U_1 + \frac{\partial U_0}{\partial Y} V_1) + \frac{\partial P_1}{\partial X} = 0, \tag{3.4b}$$

$$(U_0 \frac{\partial V_1}{\partial X} + V_0 \frac{\partial V_1}{\partial Y} + \frac{\partial V_0}{\partial X} U_1 + \frac{\partial V_0}{\partial Y} V_1) + \frac{\partial P_1}{\partial Y} = 0; \tag{3.4c}$$

$$\frac{\partial U_2}{\partial X} + \frac{\partial V_2}{\partial Y} = 0: \quad U_2 = \frac{\partial \Psi_2}{\partial Y}, \quad V_2 = -\frac{\partial \Psi_2}{\partial X}, \tag{3.5a}$$

$$\left(U_0 \frac{\partial U_2}{\partial X} + V_0 \frac{\partial U_2}{\partial Y} + \frac{\partial U_0}{\partial X} U_2 + \frac{\partial U_0}{\partial Y} V_2\right) + \frac{\partial P_2}{\partial X} = -\left(U_1 \frac{\partial U_1}{\partial X} + V_1 \frac{\partial U_1}{\partial Y}\right), \tag{3.5b}$$

$$\left(U_0 \frac{\partial V_2}{\partial X} + V_0 \frac{\partial V_2}{\partial Y} + \frac{\partial V_0}{\partial X} U_2 + \frac{\partial V_0}{\partial Y} V_2\right) + \frac{\partial P_2}{\partial Y} = -\left(U_1 \frac{\partial V_1}{\partial X} + V_1 \frac{\partial V_1}{\partial Y}\right). \tag{3.5c}$$

For these equations, the freestream boundary conditions are

$$U_0 \to 1, \; V_0, \; P_0 \to 0, \quad U_1, \; V_1, \; P_1 \to 0, \quad U_2, \; V_2, \; P_2 \to 0 \quad \text{as } Y \to \infty; \tag{3.6a}$$

the (only relevant) surface boundary condition is

$$V_0 \to 0 \quad \text{as } Y \to 0. \tag{3.6b}$$

In turn, it is possible to construct first integrals of (3.3) - (3.5). Here, the first integrals for the total pressure and the vorticity are

$$\Pi_0 = \left[P_0 + \frac{1}{2}\left(U_0^2 + V_0^2\right)\right] = B_0(\Psi_0), \tag{3.7a}$$

$$\Omega_0 = \left(\frac{\partial V_0}{\partial X} - \frac{\partial U_0}{\partial Y}\right) = -B_0'(\Psi_0); \tag{3.7b}$$

$$\Pi_1 = \left[P_1 + \left(U_0 U_1 + V_0 V_1\right)\right] = \left[\Psi_1 B_0'(\Psi_0) + B_1(\Psi_0)\right], \tag{3.8a}$$

$$\Omega_1 = \left(\frac{\partial V_1}{\partial X} - \frac{\partial U_1}{\partial Y}\right) = -\left[\Psi_1 B_0''(\Psi_0) + B_1'(\Psi_0)\right]; \tag{3.8b}$$

$$\begin{aligned}\Pi_2 &= \left[P_2 + \left(U_0 U_2 + V_0 V_2\right) + \frac{1}{2}\left(U_1^2 + V_1^2\right)\right] \\ &= \left[\Psi_2 B_0'(\Psi_0) + B_2(\Psi_0)\right] + \Psi_1\left[\frac{1}{2}\Psi_1 B_0''(\Psi_0) + B_1'(\Psi_0)\right],\end{aligned} \tag{3.9a}$$

$$\Omega_2 = \left(\frac{\partial V_2}{\partial X} - \frac{\partial U_2}{\partial Y}\right) = -\left[\Psi_2 B_0''(\Psi_0) + B_2'(\Psi_0)\right] - \Psi_1\left[\frac{1}{2}\Psi_1 B_0'''(\Psi_0) + B_1''(\Psi_0)\right]. \tag{3.9b}$$

With the freestream taken to be independent of $\delta(\varepsilon, \mu)$ [cf. (3.6a)], it is consistent to take the higher-order Bernoulli functions to be zero, that is,

$$B_1(\Psi_0),\quad B_2(\Psi_0),\ \ldots\ = 0. \tag{3.10a}$$

Further, if the exterior-region flow is irrotational (i.e., $\Omega_k = 0$, $k = 0, 1, 2, \ldots$), it follows that

$$B_0(\Psi_0) = \frac{1}{2}, \text{ and } B_0'(\Psi_0),\ B_0''(\Psi_0),\ \ldots\ = 0. \tag{3.10b}$$

Subject to the constraints of (3.10), the equations for the exterior region can be written as

$$\Omega_k = \left(\frac{\partial V_k}{\partial X} - \frac{\partial U_k}{\partial Y}\right) = -\left(\frac{\partial^2\Psi_k}{\partial X^2} + \frac{\partial^2\Psi_k}{\partial Y^2}\right) = 0 \quad (k = 0, 1, 2, \ldots); \tag{3.11a}$$

$$P_0 = \frac{1}{2}\left[1 - \left\{\left(\frac{\partial\Psi_0}{\partial X}\right)^2 + \left(\frac{\partial\Psi_0}{\partial Y}\right)^2\right\}\right],$$

$$P_1 = -\left\{\frac{\partial\Psi_0}{\partial X}\frac{\partial\Psi_1}{\partial X} + \frac{\partial\Psi_0}{\partial Y}\frac{\partial\Psi_1}{\partial Y}\right\},$$

$$P_2 = -\left[\left\{\frac{\partial\Psi_0}{\partial X}\frac{\partial\Psi_2}{\partial X} + \frac{\partial\Psi_0}{\partial Y}\frac{\partial\Psi_2}{\partial Y}\right\} + \frac{1}{2}\left\{\left(\frac{\partial\Psi_1}{\partial X}\right)^2 + \left(\frac{\partial\Psi_1}{\partial Y}\right)^2\right\}\right],\ \ldots\ . \tag{3.11b}$$

It is taken that the solutions to the differential equations, either in the forms given by (3.3) - (3.5) or in those given by (3.11), are analytic as $Y \to 0$. Thus, near the surface $(Y \to 0)$, the exterior-region-flow quantities, $G = U, V, P$, can be written as

$$\begin{aligned} G \sim {} & \left[G_0^* + \left(\frac{\partial G_0}{\partial Y}\right)^* Y + \frac{1}{2}\left(\frac{\partial^2 G_0}{\partial Y^2}\right)^* Y^2 + \ldots\right] \\ & + \delta\left[G_1 + \left(\frac{\partial G_1}{\partial Y}\right)^* Y + \ldots\right] + \delta^2\left[G_2^* + \ldots\right] + \ldots \\ = {} & \left[G_0^* + G_{01}^* Y + \frac{1}{2}G_{02}^* Y^2 + \ldots\right] + \delta\left[G_1^* + G_{11}^* Y + \ldots\right] \\ & + \delta^2\left[G_2^* + \ldots.\right] + \ldots\ , \end{aligned} \tag{3.12}$$

with G_k^*, $G_{k\ell}^*$ = fncs(X). The surface speed U_0^* arises from the basic inviscid flow [of (3.3)]; the surface boundary condition [of (3.6b)] for this basic inviscid flow gives $V_0^* = 0$. Based upon these equations of motion, it is determined that

$$U_{01}^* = 0, \; U_{02}^* = -\frac{d^2U_0^*}{dX^2}, \; \ldots , \; U_{11}^* = \frac{dV_1^*}{dX}, \; \ldots ; \tag{3.13a}$$

$$V_{01}^* = -\frac{dU_0^*}{dX}, \; V_{02}^* = 0, \; \ldots , \; V_{11}^* = -\frac{dU_1^*}{dX}, \; \ldots ; \tag{3.13b}$$

$$P_0^* = \frac{1}{2}\,(1-U_0^{*2}), \; P_{01}^* = 0, \; \ldots , \; P_1^* = -\,U_0^*\,U_1^*, \; \ldots . \tag{3.13c}$$

4. THE TURBULENT BOUNDARY LAYER

For the turbulent boundary layer, the coordinates are

$$x = X, \quad y = Y/\delta, \text{ with } \delta \to 0; \tag{4.1}$$

the flow quantities are

$$U = u, \ V = \delta v, \ P = p, \ T_{ij} = \tau_{ij} . \tag{4.2}$$

Here, $\delta = \delta(\varepsilon,\mu)$; $G = G(X,Y;\ \varepsilon,\mu) = G(X,Y;\ \delta)$, with $G = U, V, P, T_{ij}$; $g = g(x,y;\ \delta)$, with $g = u, v, p, \tau_{ij}$. Introduction of these boundary-layer variables into (2.1) and (2.2) yields

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0: \quad u = \frac{\partial \psi}{\partial y}, \ v = -\frac{\partial \psi}{\partial x} ; \tag{4.3}$$

$$\left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}\right) + \frac{\partial p}{\partial x} = \left(\frac{\varepsilon^2}{\delta}\right)\left(\frac{\partial \tau_{xy}}{\partial y} + \delta \frac{\partial \tau_{xx}}{\partial x}\right) + \left(\frac{\mu}{\delta^2}\right)\left(\frac{\partial^2 u}{\partial y^2} + \delta^2 \frac{\partial^2 u}{\partial x^2}\right) , \tag{4.4a}$$

$$\delta^2\left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y}\right) + \frac{\partial p}{\partial y} = \delta\left(\frac{\varepsilon^2}{\delta}\right) \left(\frac{\partial \tau_{yy}}{\partial y} + \delta \frac{\partial \tau_{yx}}{\partial x}\right) + \delta^2\left(\frac{\mu}{\delta^2}\right)\left(\frac{\partial^2 v}{\partial y^2} + \delta^2 \frac{\partial^2 v}{\partial x^2}\right). \tag{4.4b}$$

The surface boundary conditions are

$$u, \ v \to 0, \quad \tau_{ij} \to 0 \text{ as } y \to 0. \tag{4.5a}$$

The interior-region--exterior-region matching conditions are

$$u \to U, \ v \to V/\delta, \ p \to P, \ \tau_{ij} \to T_{ij} \sim 0 \text{ as } y \to \infty, \ Y \to 0. \tag{4.5b}$$

4.1 The Outer Layer

The analysis of the boundary layer is initiated with the consideration of an outer layer, defined by x, y fixed, with ε, μ, and $\delta(\varepsilon,\mu) \to 0$. For this

outer layer, it is taken that, to leading order of approximation, the outer-layer x-momentum equation, (4.4a), is characterized by a convection--pressure-gradient--turbulent-stress balance (with the viscous stress negligible). Specifically, it is taken that the outer-layer thickness parameter is equal to the square of the turbulent-velocity parameter, that is,

$$\delta(\varepsilon,\mu) = \delta(\varepsilon(\mu)) = \varepsilon^2(\mu) \to 0: \quad \varepsilon(\delta) = \delta^{1/2} \to 0, \tag{4.6}$$

with $\mu(\delta) = \delta^{7/2} \to 0$ (to be justified in Section 4.2), such that $(\mu/\delta^2) = \delta^{3/2} \to 0$. For these orderings of the parameters, (4.4a) and 4.4b), employing the thickness parameter, δ, throughout, take the forms

$$\left(u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y}\right) + \frac{\partial p}{\partial x} = \left(\frac{\partial \tau_{xy}}{\partial y} + \delta \frac{\partial \tau_{xx}}{\partial x}\right) + \delta^{3/2} \left(\frac{\partial^2 u}{\partial y^2} + \delta^2 \frac{\partial^2 u}{\partial x^2}\right) , \tag{4.4a$'$}$$

$$\delta^2\left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y}\right) + \frac{\partial p}{\partial y} = \delta \left(\frac{\partial \tau_{yy}}{\partial y} + \delta \frac{\partial \tau_{yx}}{\partial x}\right) + \delta^{7/2} \left(\frac{\partial^2 v}{\partial y^2} + \delta^2 \frac{\partial^2 v}{\partial x^2}\right) . \tag{4.4b$'$}$$

In turn, from an examination of (4.4)', it is taken that the outer-layer expansions for the dependent variables are of the form

$$g(x,y;\delta) \cong g_0(x,y) + \delta^{1/2}g_1(x,y) + \delta g_2(x,y) + \delta^{3/2}g_3(x,y) + \dots , \tag{4.7}$$

where $g = u, v, p, \tau_{ij}$.

Thus, the leading-order approximations for the outer layer are

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0: \quad u = \frac{\partial \phi_0}{\partial y}, \ v_0 = - \frac{\partial \phi_0}{\partial x} ,$$

$$\left(u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial y}\right) + \frac{\partial p_0}{\partial x} - \frac{\partial \tau_{xy0}}{\partial y} = 0, \quad \frac{\partial p_0}{\partial y} = 0; \tag{4.8a-c}$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0: \quad u_1 = \frac{\partial \phi_1}{\partial y}, \ v_1 = - \frac{\partial \phi_1}{\partial x} ,$$

$$\left(u_0 \frac{\partial u_1}{\partial x} + v_0 \frac{\partial u_1}{\partial y} + \frac{\partial u_0}{\partial x} u_1 + \frac{\partial u_0}{\partial y} v_1\right) + \frac{\partial p_1}{\partial x} - \frac{\partial \tau_{xy1}}{\partial y} = 0, \quad \frac{\partial p_1}{\partial y} = 0; \tag{4.9a-c}$$

$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = 0: \quad u = \frac{\partial \psi_2}{\partial y}, \; v_2 = - \frac{\partial \psi_2}{\partial x} ,$$

$$(u_0 \frac{\partial u_2}{\partial x} + v_0 \frac{\partial u_2}{\partial y} + \frac{\partial u_0}{\partial x} u_2 + \frac{\partial u_0}{\partial y} v_2) + \frac{\partial p_2}{\partial x} - \frac{\partial \tau_{xy2}}{\partial y}$$

$$= \frac{\partial \tau_{xx0}}{\partial x} - (u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y}) ,$$

$$\frac{\partial p_2}{\partial y} = \frac{\partial \tau_{yy0}}{\partial y}; \tag{4.10a-c}$$

$$\frac{\partial u_3}{\partial x} + \frac{\partial v_3}{\partial y} = 0: \quad u_3 = \frac{\partial \psi_3}{\partial y}, \; v_3 = - \frac{\partial \psi_3}{\partial x},$$

$$(u_0 \frac{\partial u_3}{\partial x} + v_0 \frac{\partial u_3}{\partial y} + \frac{\partial u_0}{\partial x} u_3 + \frac{\partial u_0}{\partial y} v_3) + \frac{\partial p_3}{\partial x} - \frac{\partial \tau_{xy3}}{\partial y}$$

$$= \frac{\partial^2 u_0}{\partial y^2} + \frac{\partial \tau_{xx1}}{\partial x} - (u_1 \frac{\partial u_2}{\partial x} + v_1 \frac{\partial u_2}{\partial y} + \frac{\partial u_1}{\partial x} u_2 + \frac{\partial u_1}{\partial y} v_2),$$

$$\frac{\partial p_3}{\partial y} = \frac{\partial \tau_{yy1}}{\partial y} . \tag{4.11a-c}$$

In what follows, the notation $\tau_{xy} = \tau_{yx} = \tau$, and $\tau_{xyk} = \tau_{yxk} = \tau_k$, $k = 0, 1, 2, 3, \ldots$, is employed.

In this paper, for specificity, the mixing-length-theory closure is adopted for the nearfield behavior of the outer layer. That is, the turbulent length-scale function, ℓ, is taken to be

$$\ell = \tau^{1/2}(\frac{\partial u}{\partial y})^{-1} \to \kappa y \text{ as } y \to 0, \text{ with } \kappa = \text{von Karman const.} \tag{4.12}$$

In terms of the outer-layer expansions, this nearfield behavior ($y \to 0$) can be expressed as

$$\tau_0^{1/2}(\frac{\partial u_0}{\partial y})^{-1} \to \kappa y:$$

$$\tau_0 \to \kappa^2 y^2 (\frac{\partial u_0}{\partial y})^2 \; ; \tag{4.8d}$$

$$[\frac{1}{2}(\frac{\tau_1}{\tau_0}) - (\frac{\partial u_1/\partial y}{\partial u_0/\partial y})] \to 0:$$

$$\tau_1 \to 2\kappa^2 y^2 \frac{\partial u_0}{\partial y}\frac{\partial u_1}{\partial y} ; \tag{4.9d}$$

$$[\frac{1}{2}(\frac{\tau_2}{\tau_0}) - (\frac{\partial u_2/\partial y}{\partial u_0/\partial y}) - \frac{1}{2}(\frac{\partial u_1/\partial y}{\partial u_0/\partial y})^2] \to 0:$$

$$\tau_2 \to 2\kappa^2 y^2 [\frac{\partial u_0}{\partial y}\frac{\partial u_2}{\partial y} + \frac{1}{2}(\frac{\partial u_1}{\partial y})^2] ; \tag{4.10d}$$

$$[\frac{1}{2}(\frac{\tau_3}{\tau_0}) - (\frac{\partial u_3/\partial y}{\partial u_0/\partial y}) - (\frac{\partial u_1/\partial y}{\partial u_0/\partial y})(\frac{\partial u_2/\partial y}{\partial u_0/\partial y}) + (\frac{\partial u_1/\partial y}{\partial u_0/\partial y})^3] \to 0:$$

$$\tau_3 \to 2\kappa^2 y^2 [\frac{\partial u_0}{\partial y}\frac{\partial u_3}{\partial y} + \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial y} - (\frac{\partial u_0}{\partial y})^{-1}(\frac{\partial u_1}{\partial y})^3]. \tag{4.11d}$$

In the same spirit, the farfield behavior of the outer-layer eddy viscosity, ν_e, is taken to be

$$\nu_e = \tau(\frac{\partial u}{\partial y})^{-1} \to k \text{ as } y \to \infty, \tag{4.13}$$

with k = fnc(x) (to be specified). In terms of the outer-layer expansions, this farfield behavior ($y \to \infty$) can be expressed as

$$\tau_0 \to k \frac{\partial u_0}{\partial y} ; \tag{4.8e}$$

$$\tau_1 \to k \frac{\partial u_1}{\partial y} ; \tag{4.9e}$$

$$\tau_2 \to k \frac{\partial u_2}{\partial y} ; \tag{4.10e}$$

$$\tau_3 \to k \frac{\partial u_3}{\partial y} . \tag{4.11e}$$

Thus, the zeroth-order approximation for the outer layer is

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0: \quad u_0 = \frac{\partial \psi_0}{\partial y}, \; v_0 = -\frac{\partial \psi_0}{\partial x} , \tag{4.14a}$$

$$(u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial y}) + \frac{\partial p_0}{\partial x} - \frac{\partial \tau_0}{\partial y} = 0, \quad \frac{\partial p_0}{\partial y} = 0 , \tag{4.14b,c}$$

The outer-edge ($y \to \infty$) boundary conditions are

$$u_0 \sim \overset{*}{u}_0 + \dots , \quad v_0 \sim -\frac{d\overset{*}{u}_0}{dx} y + \overset{*}{v}_0 + \dots ,$$

$$p_0 \sim \overset{*}{p}_0 + \dots , \quad \tau_0 \sim k \frac{\partial u_0}{\partial y} \sim 0 , \tag{4.15}$$

where $\overset{*}{u}_0$ and $\overset{*}{p}_0$ are the externally applied streamwise velocity and pressure, respectively. Based on an external flow that is irrotational (cf. Section 3), it follows that, to leading order of approximation, $\overset{*}{p}_0 = \frac{1}{2}(1-\overset{*}{u}_0{}^2)$, such that $(d\overset{*}{p}_0/dx) = -\overset{*}{u}_0(d\overset{*}{u}_0/dx) > 0$ for the adverse-pressure-gradient case under consideration here. The (only relevant) surface ($y \to 0$) boundary condition is

$$v_0 \sim 0. \tag{4.16}$$

Directly, it is seen, from (4.14c), that $p_0 = \overset{*}{p}_0 = \text{fnc}(x)$. In turn, (4.14b) becomes

$$(u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial y}) + \frac{d\overset{*}{p}_0}{dx} - \frac{\partial \tau_0}{\partial y} = 0 . \tag{4.14b)'$$

Based upon the preceding, the following integral relations are developed:

$$\overset{*}{v}_0 = \frac{d(\overset{*}{u}_0\overset{*}{\Delta}_0)}{dx} , \tag{4.17a}$$

$$\tau_0^+ = \overset{*}{u}_0{}^2 [\frac{d\overset{*}{\Theta}_0}{dx} + \frac{1}{\overset{*}{u}_0} \frac{d\overset{*}{u}_0}{dx} (2\overset{*}{\Theta}_0 + \overset{*}{\Delta}_0)] . \tag{4.17b}$$

for $\tau_0 \to \tau_0^+$ as $y \to 0$, with

$$\Delta_0^* = \int_0^\infty (1 - \frac{u_0}{u_0^*})\, dy, \quad \theta_0^* = \int_0^\infty \frac{u_0}{u_0^*}(1 - \frac{u_0}{u_0^*})\, dy, \tag{4.18a,b}$$

the zeroth-order defect and momentum thicknesses, respectively.

Based on the results of Stratford (1959a), it is anticipated that, as $y \to 0$, $u_0 \propto y^{1/2} \to 0$ and $\tau_0 \propto y \to 0$. Specifically, the solutions of (4.14a) and (4.14b)', as $y \to 0$, taking into account the nearfield closure hypothesis of (4.8d), are

$$u_0 \sim \frac{2}{\kappa}(\frac{dp_0^*}{dx})^{1/2} y^{1/2} \, [1 + \frac{1}{9\kappa^2}(\frac{dp_0^*}{dx})^{-1} \frac{d^2p_0^*}{dx^2} y + \ldots] \to 0, \tag{4.19a}$$

$$v_0 \sim -\frac{2}{3\kappa}(\frac{dp_0^*}{dx})^{-1/2} \frac{d^2p_0^*}{dx^2} y^{3/2} \, [1 + \ldots] \to 0\ , \tag{4.19b}$$

$$\frac{\partial u_0}{\partial y} \sim \frac{1}{\kappa}(\frac{dp_0^*}{dx})^{1/2} y^{-1/2} \, [1 + \frac{1}{3\kappa^2}(\frac{dp_0^*}{dx})^{-1} \frac{d^2p_0^*}{dx^2} y + \ldots] \to \infty\ , \tag{4.19c}$$

$$\tau_0 \sim \frac{dp_0^*}{dx} y \, [1 + \frac{2}{3\kappa^2}(\frac{dp_0^*}{dx})^{-1} \frac{d^2p_0^*}{dx^2} y + \ldots] \to 0. \tag{4.19d}$$

These inner-edge ($y, \eta \to 0$) results may be expressed as

$$\bar{u}_0 = (\frac{u_0}{u_0^*}) \sim \frac{2}{\kappa}\eta^{1/2} \, [1 - \frac{A}{9\kappa^2}\eta + \ldots], \tag{4.20a}$$

$$\bar{v}_0 = (\frac{v_0}{u_0^*}) \sim \frac{2A}{3K}\eta^{3/2} \, [1 + \ldots]\ , \tag{4.20b}$$

$$\frac{\partial \bar{u}_0}{\partial \eta} = \frac{\partial(u_0/u_0^*)}{\partial(y/h_0^*)} \sim \frac{1}{\kappa}\eta^{-1/2} \, [1 - \frac{A}{3\kappa^2}\eta + \ldots]\ , \tag{4.20c}$$

$$\bar{\tau}_0 = \left(\frac{\tau_0}{u_0^{*2}}\right) \sim \eta \left[1 - \frac{2A}{3\kappa^2}\eta + \ldots\right] . \tag{4.20d}$$

Here, $u_0^* = (1-2p_0^*)^{1/2}$, and

$$\eta = \left(\frac{y}{h_0^*}\right), \text{ with } h_0^* = \left[(1-2p_0^*)\left(\frac{dp_0^*}{dx}\right)^{-1}\right] , \tag{4.21}$$

$$A = \left[(1-2p_0^*)\left(\frac{dp_0^*}{dx}\right)^{-2}\left(-\frac{d^2p_0^*}{dx^2}\right)\right]. \tag{4.22}$$

Further, the outer-edge ($y,\eta \to \infty$) results may now be expressed as

$$\bar{u}_0 \to 1, \tag{4.23a}$$

$$\bar{v}_0 \to \eta + \frac{1}{u_0^*}\frac{d(u_0^* h_0^* \Delta_\eta)}{dx}, \text{ with } \Delta_\eta = \left(\frac{\Delta_0^*}{h_0^*}\right) , \tag{4.23b}$$

$$\frac{\partial \bar{u}_0}{\partial \eta} \to 0, \ \bar{\tau}_0 \to 0:$$

$$\bar{\tau}_0 \left(\frac{\partial \bar{u}_0}{\partial \eta}\right)^{-1} \to K\Delta_\eta, \text{ for } k = Ku_0^*\Delta_0^*, \text{ with } K = \text{Clauser const.} \tag{4.23c,d}$$

4.2 The Inner Layer

The results of (4.19) and (4.20) indicate that the outer-layer flow variables u_0 and $\tau_0 \to 0$ and $(\partial u_0/\partial y) \to \infty$ in the limit of $y \to 0$, $\delta \to 0$; however, it is seen that $u_0/\delta^{1/2}$ and τ_0/δ are fixed for $r = y/\delta$ fixed. Based on these results, it is appropriate to introduce a layer interior to the outer layer. It is in this inner layer that the surface boundary conditions are satisfied uniformly. Based on the aforementioned results, this inner layer is defined by

$$x = X,\ r = y/\delta = Y/\delta^2,\ \text{with } \delta \to 0, \tag{4.24}$$

with the flow quantities given by

$$U = u = \delta^{1/2}m,\ V = \delta v = \delta^{5/2}n,\ P = p = q,\ T_{ij} = \tau_{ij} = \delta t_{ij}\ . \tag{4.25}$$

Here, $G = G(X,Y;\delta)$, with $G = U, V, P, T_{ij}$; $g = g(x,y;\delta)$, with $g = u, v, p, \tau_{ij}$; $f = f(x,r;\delta)$, with $f = m, n, q, t_{ij}$. In terms of the inner-layer variables, the equations of motion are

$$\frac{\partial m}{\partial x} + \frac{\partial n}{\partial r} = 0: \quad m = \frac{\partial \phi}{\partial r},\ n = -\frac{\partial \phi}{\partial x}\ ; \tag{4.26}$$

$$\delta(m\frac{\partial m}{\partial x} + n\frac{\partial m}{\partial r}) + \frac{\partial q}{\partial x} = (\frac{\partial t_{xr}}{\partial r} + \delta^2\frac{\partial t_{xx}}{\partial x}) + (\frac{\mu}{\delta^{7/2}})(\frac{\partial^2 m}{\partial r^2} + \delta^4\frac{\partial^2 m}{\partial x^2})\ , \tag{4.27a}$$

$$\delta^5(m\frac{\partial n}{\partial x} + n\frac{\partial n}{\partial r}) + \frac{\partial q}{\partial r} = \delta^2(\frac{\partial t_{rr}}{\partial r} + \delta^2\frac{\partial t_{rx}}{\partial x}) + \delta^4(\frac{\mu}{\delta^{7/2}})(\frac{\partial^2 n}{\partial r^2} + \delta^4\frac{\partial^2 n}{\partial x^2}). \tag{4.27b}$$

In order that, to leading order of approximation, the viscous-stress contribution be of the same order of magnitude as the turbulent-stress and pressure-gradient contributions in the x-momentum equation, (4.27a), it is necessary that

$$\mu(\delta) = \delta^{7/2} \to 0 \text{ as } \delta \to 0, \tag{4.28}$$

as well as $\varepsilon(\delta) = \delta^{1/2} \to 0$ [see (4.6)]. Thus, since $\mu = R^{-1}$, where $R = (\tilde{u}_\infty\tilde{c}/\tilde{\nu})$ is the reference Reynolds number, it is determined that the present analysis holds for

$$\delta(R) = R^{-2/7} \to 0,\ \varepsilon(R) = R^{-1/7} \to 0 \text{ as } R \to \infty. \tag{4.29a,b}$$

For this ordering of the parameters, (4.27a,b) can be written as

$$\delta(m\frac{\partial m}{\partial x} + n\frac{\partial m}{\partial r}) + \frac{\partial q}{\partial x} = (\frac{\partial t_{xr}}{\partial r} + \delta^2\frac{\partial t_{xx}}{\partial x}) + (\frac{\partial^2 m}{\partial r^2} + \delta^4\frac{\partial^2 m}{\partial x^2})\ , \tag{4.27a'}$$

$$\delta^5\left(m\frac{\partial n}{\partial x} + n\frac{\partial n}{\partial r}\right) + \frac{\partial q}{\partial r} = \delta^2\left(\frac{\partial t_{rr}}{\partial r} + \delta^2\frac{\partial t_{rx}}{\partial x}\right) + \delta^4\left(\frac{\partial^2 n}{\partial r^2} + \delta^4\frac{\partial^2 n}{\partial x^2}\right) . \qquad (4.27b)'$$

From an examination of (4.26) and (4.27)', the inner-layer expansions for the dependent variables are

$$f(x,r;\delta) \cong f_0(x,r) + \delta^{1/2}f_1(x,r) + \delta f_2(x,r) + \delta^{3/2}f_3(x,r) + \ldots , \qquad (4.30)$$

where $f = m,\ n,\ q,\ t_{ij}$.

The resulting equations for the leading-order approximation for this inner layer are

$$\frac{\partial m_0}{\partial x} + \frac{\partial n_0}{\partial r} = 0:\quad m_0 = \frac{\partial \phi_0}{\partial r},\quad n_0 = -\frac{\partial \phi_0}{\partial x},$$

$$\frac{\partial t_{xr0}}{\partial r} + \frac{\partial^2 m_0}{\partial r^2} - \frac{\partial q_0}{\partial x} = 0,\quad \frac{\partial q_0}{\partial r} = 0; \qquad (4.31a\text{-}c)$$

$$\frac{\partial m_1}{\partial x} + \frac{\partial n_1}{\partial r} = 0:\quad m_1 = \frac{\partial \phi_1}{\partial r},\quad n_1 = -\frac{\partial \phi_1}{\partial x},$$

$$\frac{\partial t_{xr1}}{\partial r} + \frac{\partial^2 m_1}{\partial r^2} - \frac{\partial q_1}{\partial x} = 0,\quad \frac{\partial q_1}{\partial r} = 0; \qquad (4.32a\text{-}c)$$

$$\frac{\partial m_2}{\partial x} + \frac{\partial n_2}{\partial r} = 0:\quad m_2 = \frac{\partial \phi_2}{\partial r},\quad n_2 = -\frac{\partial \phi_2}{\partial x},$$

$$\frac{\partial t_{xr2}}{\partial r} + \frac{\partial^2 m_2}{\partial r^2} - \frac{\partial q_2}{\partial x} = \left(m_0\frac{\partial m_0}{\partial x} + n_0\frac{\partial m_0}{\partial r}\right),\quad \frac{\partial q_2}{\partial r} = 0; \qquad (4.33a\text{-}c)$$

$$\frac{\partial m_3}{\partial x} + \frac{\partial n_3}{\partial r} = 0:\quad m_3 = \frac{\partial \phi_3}{\partial r},\quad n_3 = -\frac{\partial \phi_3}{\partial x},$$

$$\frac{\partial t_{xr3}}{\partial r} + \frac{\partial^2 m_3}{\partial r^2} - \frac{\partial q_3}{\partial x} = \left(m_0\frac{\partial m_1}{\partial x} + n_0\frac{\partial m_1}{\partial r} + \frac{\partial m_0}{\partial x}m_1 + \frac{\partial m_0}{\partial r}n_1\right),$$

$$\frac{\partial q_3}{\partial r} = 0 . \qquad (4.34a\text{-}c)$$

In what follows, the notation $t_{xr} = t_{rx} = t$ and $t_{xrk} = t_{rxk} = t_k$, $k = 0, 1, 2, 3, \ldots$, is employed.

Thus, the equations for the zeroth-order approximation are

$$\frac{\partial m_0}{\partial x} + \frac{\partial n_0}{\partial r} = 0: \quad m_0 = \frac{\partial \phi_0}{\partial r}, \quad n_0 = -\frac{\partial \phi_0}{\partial x} \tag{4.35}$$

$$\frac{\partial t_0}{\partial r} + \frac{\partial^2 m_0}{\partial r^2} - \frac{\partial q_0}{\partial x} = 0, \quad \frac{\partial q_0}{\partial r} = 0 . \tag{4.36a,b}$$

The surface ($r \to 0$) boundary conditions are

$$m_0, n_0 \to 0, \; t_0 \to 0. \tag{4.37}$$

In this approximation, from (4.36b), the pressure is found to be

$$q_0 = q_0^0 = \mathrm{fnc}(x) . \tag{4.38}$$

Directly, the first integral of the x-momentum equation, (4.36a), is determined to be

$$t_0 + \frac{\partial m_0}{\partial r} - \frac{dq_0^0}{dx} r = \left(\frac{\partial m_0}{\partial r}\right)^0 = \mathrm{fnc}(x) . \tag{4.39}$$

For the inner layer, the appropriate turbulent length-scale function is λ, defined by

$$\lambda = (\ell/\delta) = t^{1/2} \left(\frac{\partial m}{\partial r}\right)^{-1} .$$

$$\cong \lambda_0 + \ldots = t_0^{1/2} \left(\frac{\partial m_0}{\partial r}\right)^{-1} + \ldots . \tag{4.40}$$

At the outer edge ($r \to \infty$) of the inner layer, the mixing-length-theory closure of $\lambda_0 \to \kappa r \to \infty$ holds. This behavior is consistent with the previously adopted inner-edge behavior of the outer-layer length-scale function [i.e., $\ell \sim \kappa y \to 0$

as $y \to 0$]. For $\lambda_0/\kappa r \to 1$ (exponentially) as $r \to \infty$, from (4.39) and (4.40), it follows that, with $q_0^+ = q_0^0$ and $t_0^+ = (\partial m_0/\partial r)^0$,

$$\frac{1}{\kappa r}\left(\frac{dq_0^+}{dx} r + t_0^+ - \frac{\partial m_0}{\partial r}\right)^{1/2} \left(\frac{\partial m_0}{\partial r}\right)^{-1} \to 1 \quad \text{(exponentially)} \ . \tag{4.41}$$

From this relation, it is determined that, in this limit,

$$\frac{\partial m_0}{\partial r} \sim \frac{1}{\kappa}\left(\frac{dq_0^+}{dx}\right)^{1/2} r^{-1/2} + \frac{1}{2\kappa} t_0^+ \left(\frac{dq_0^+}{dx}\right)^{-1/2} r^{-3/2} - \frac{1}{2\kappa^2} r^{-2} + \ldots \to 0. \tag{4.42a}$$

Integration produces

$$m_0 \sim \frac{2}{\kappa}\left(\frac{dq_0^+}{dx}\right)^{1/2} r^{1/2} + m_0^+ - \frac{1}{\kappa} t_0^+ \left(\frac{dq_0^+}{dx}\right)^{-1/2} r^{-1/2} + \frac{1}{2\kappa^2} r^{-1} + \ldots \to \infty, \tag{4.42b}$$

where m_0^+, ... = fncs(x) (to be determined). In turn,

$$t_0 \sim \frac{dq_0^+}{dx} r + t_0^+ - \frac{1}{\kappa}\left(\frac{dq_0^+}{dx}\right)^{1/2} r^{-1/2} + \ldots \to \infty. \tag{4.43}$$

These outer-edge results may also be expressed as

$$\bar{m}_0 = m_0 \left(\frac{dq_0^+}{dx}\right)^{-1/3} \sim \frac{2}{\kappa}\zeta^{1/2} + \gamma^+ - \frac{\omega^+}{\kappa}\zeta^{-1/2} + \frac{1}{2\kappa^2}\zeta^{-1} + \ldots \ , \tag{4.44a}$$

$$\bar{t}_0 = t_0 \left(\frac{dq_0^+}{dx}\right)^{-2/3} \sim \zeta + \omega^+ - \frac{1}{\kappa}\zeta^{-1/2} + \ldots \ , \tag{4.44b}$$

$$\bar{\lambda}_0 = \lambda_0 \left(\frac{dq_0^+}{dx}\right)^{1/3} \sim \kappa\zeta(1 + \ldots) \ , \tag{4.44c}$$

where

$$\zeta = r\left(\frac{dq_0^+}{dx}\right)^{1/3}, \tag{4.45a}$$

$$\gamma^+ = m_0^+\left(\frac{dq_0^+}{dx}\right)^{-1/3}, \quad \omega^+ = t_0^+\left(\frac{dq_0^+}{dx}\right)^{-2/3}, \; \ldots \; . \tag{4.45b}$$

4.3 Matching of Zeroth-Order Solutions

A comparison of (4.19), (4.20), and (4.42), (4.43) indicates that the zeroth-order outer-layer and inner-layer solutions for the streamwise velocity and Reynolds stress match if $q_0^+ = p_0^* = \frac{1}{2}(1-u_0^{*2})^{1/2}$, that is,

$$u \sim \left(\left[\frac{2}{\kappa}\left(\frac{dp_0^*}{dx}\right)^{1/2} y^{1/2} + \ldots\right] + \ldots\right)$$

$$\sim \delta^{1/2}\left(\left[\frac{2}{\kappa}\left(\frac{dp_0^*}{dx}\right)^{1/2} r^{1/2} + \ldots\right] + \ldots\right); \tag{4.46}$$

$$\tau \sim \left(\left[\frac{dp_0^*}{dx}\, y + \ldots\right] + \ldots\right) \sim \delta\left(\left[\frac{dp_0^*}{dx}\, r + \ldots\right] + \ldots\right). \tag{4.47}$$

Further consideration of the matching of the outer-layer and inner-layer solutions yields

$$u \sim \left[\frac{2}{\kappa}\left(\frac{dp_0^*}{dx}\right)^{1/2} y^{1/2} + \frac{2}{9\kappa^3}\left(\frac{dp_0^*}{dx}\right)^{-1/2}\frac{d^2p_0^*}{dx^2}\, y^{3/2} + \ldots\right]$$

$$+ \delta^{1/2}\left[m_0^+ + \ldots\right] + \delta\left[-\frac{1}{\kappa} t_0^+\left(\frac{dp_0^*}{dx}\right)^{-1/2} y^{1/2} + \ldots\right]$$

$$+ \delta^{3/2}\left[\frac{1}{2\kappa^2} y^{-1} + \ldots\right] + \ldots , \tag{4.48a}$$

$$\tau \sim [\frac{dp_0^*}{dx} y + \frac{2}{3\kappa^2}(\frac{d^2p_0^*}{dx^2}) y^2 + \dots]$$

$$+ \delta^{1/2} [\dots] + \delta [t_0^+ + \dots]$$

$$+ \delta^{3/2} [- \frac{1}{\kappa}(\frac{dp_0^*}{dx})^{1/2} y^{-1/2} + \dots] + \dots ; \qquad (4.48b)$$

$$m \sim [\frac{2}{\kappa}(\frac{dp_0^*}{dx})^{1/2} r^{1/2} + m_0^+ - \frac{1}{\kappa} t_0^+ (\frac{dp_0^*}{dx})^{-1/2} r^{-1/2} + \frac{1}{2\kappa^2} r^{-1} + \dots]$$

$$+ \delta^{1/2} [\dots] + \delta [\frac{2}{9\kappa^3}(\frac{dp_0^*}{dx})^{-1/2} \frac{d^2p_0^*}{dx^2} r^{3/2} + \dots] + \dots , \qquad (4.49a)$$

$$t \sim [\frac{dp_0^*}{dx} r + t_0^+ - \frac{1}{\kappa}(\frac{dp_0^*}{dx})^{1/2} r^{-1/2} + \dots]$$

$$+ \delta^{1/2} [\dots] + \delta [\frac{2}{3\kappa^2}\frac{d^2p_0^*}{dx^2} r^2 + \dots] + \dots . \qquad (4.49b)$$

5. ZEROTH-ORDER-APPROXIMATION SIMILARITY FORMULATIONS

5.1 The Zeroth-Order Approximation for the Outer Layer

The results of (4.20) - (4.23) suggest that a similarity formulation for this zeroth-order approximation exists. For this similarity formulation, with $u_0^* = [(1-2p_0^*)^{1/2}]$, $h_0^* = [(1-2p_0^*)(dp_0^*/dx)^{-1}]$, the independent variables are

$$(x,\eta), \text{ with } \eta = \eta(x,y) = \frac{y}{h_0^*(x)} = [(1-2p_0^*)^{-1} \frac{dp_0^*}{dx}]\, y \; ; \tag{5.1}$$

and the dependent variables are

$$\psi_0(x,y) = u_0^*(x)h_0^*(x)f_0(\eta) = [(1-2p_0^*)^{3/2} (\frac{dp_0^*}{dx})^{-1}]\, f_0(\eta);$$

$$u_0 = \frac{\partial\psi_0}{\partial y} = u_0^* \frac{df_0}{d\eta} = (1-2p_0^*)^{1/2} \frac{df_0}{d\eta} ,$$

$$v_0 = -\frac{\partial\psi_0}{\partial x} = (1-2p_0^*)^{1/2} [(A-2)\eta \frac{df_0}{d\eta} - (A-3)f_0] , \tag{5.2a}$$

$$\tau_0(x,y) = u_0^{*2}(x)\phi_0(\eta) = (1-2p_0^*)\phi_0(\eta) . \tag{5.2b}$$

From the nearfield mixing-length hypothesis, similarity requires

$$A = [(1-2p_0^*)(\frac{dp_0^*}{dx})^{-2} (-\frac{d^2p_0^*}{dx^2})] = \text{const.} \tag{5.3}$$

This requirement with respect to A indicates that similarity holds only for a specific class of applied pressure distributions, $p_0^*(x)$. More will be said about this later.

Introduction of these similarity variables into the equations of motion for this zeroth approximation produces

$$\frac{d\phi_0}{d\eta} = [1 - (\frac{df_0}{d\eta})^2] - (A-3) f_0 \frac{d^2 f_0}{d\eta^2} . \qquad (5.4)$$

The farfield boundary conditions are

$$\frac{df_0}{d\eta} \to 1, \ \phi_0 \to K\Delta_\eta \frac{d^2 f_0}{d\eta^2} \to 0 \text{ as } \eta \to \infty ; \qquad (5.5)$$

the nearfield boundary conditions are

$$\frac{df_0}{d\eta} \sim \frac{2}{\kappa} \eta^{1/2} [1 - \frac{A}{9\kappa^2} \eta + \ldots] \to 0,$$

$$\phi_0 \sim [\kappa\eta \frac{d^2 f_0}{d\eta^2}]^2 \sim \eta [1 - \frac{2A}{3\kappa^2} \eta + \ldots] \to 0 \text{ as } \eta \to 0. \qquad (5.6)$$

Integration of (5.4) over the domain of η, subject to (5.5) and (5.6), yields

$$\Delta_\eta - (A-4)\, \Theta_\eta = 0: \ \ A = (H_0 + 4) , \qquad (5.7a)$$

where

$$\Delta_\eta = (\frac{\Delta_0^*}{h_0^*}) = \int_0^\infty (1 - \frac{df_0}{d\eta})\, d\eta, \ \Theta_\eta = (\frac{\Theta_0^*}{h_0^*}) = \int_0^\infty \frac{df_0}{d\eta} (1 - \frac{df_0}{d\eta})\, d\eta:$$

$$H_0 = \frac{\Delta_\eta}{\Theta_\eta} . \qquad (5.7b)$$

Thus, for this similarity formulation, the shape factor, H_0, is related to the applied pressure distribution, p_0^*, by

$$H_0 = [(1-2p_0^*)(\frac{dp_0^*}{dx})^{-2} (- \frac{d^2 p_0^*}{dx^2}) - 4] = \text{const.} \qquad (5.8)$$

The solution of this equation for p_0^* is

$$p_0^* = \frac{1}{2}\{1 - [C + (H_0 + 2)Bx]^{-2/(H_0+2)}\} \ , \tag{5.9}$$

where B and C are constants of integration. Quantities associated with this solution are

$$u_0^* = (1-2p_0^*)^{1/2} = [C + (H_0 + 2)Bx]^{-1/(H_0+2)} \tag{5.10a}$$

$$\frac{dp_0^*}{dx} = B[C + (H_0 + 2)Bx]^{-(H_0+4)/(H_0+2)} \ , \tag{5.10b}$$

$$h_0^* = [(1-2p_0^*)(\frac{dp_0^*}{dx})^{-1}] = \frac{1}{B}[C + (H_0 + 2)Bx] \ . \tag{5.10c}$$

With the appropriate choice of the constants B and C, the results of (5.10) can be written as

$$h_0^* = (H_0 + 2)(x - x_0) \ , \tag{5.11a}$$

$$u_0^* = u_{00}^* (x - x_0)^{-1/(H_0+2)} \ , \tag{5.11b}$$

$$\frac{dp_0^*}{dx} = \frac{u_{00}^{*2}}{(H_0+2)}(x - x_0)^{-(H_0+4)/(H_0+2)} \ . \tag{5.11c}$$

In terms of H_0, rather than A, the similarity boundary-value problem for the zeroth-order approximation for the outer layer is

$$\frac{d\phi_0}{d\eta} = [1 - (\frac{df_0}{d\eta})^2] - (H_0 + 1) f_0 \frac{d^2f_0}{d\eta^2} \ ; \tag{5.12}$$

$$\frac{df_0}{d\eta} \rightarrow 1, \ \phi_0 \rightarrow (K\Delta_\eta) \frac{d^2f_0}{d\eta^2} \rightarrow 0 \text{ as } \eta \rightarrow \infty, \tag{5.13a}$$

$$\frac{df_0}{d\eta} - \beta_{01}^+\eta^{1/2} + \beta_{03}^+\eta^{3/2} + \beta_{05}^+\eta^{5/2} + \dots \rightarrow 0,$$

$$\text{with } \beta_{01}^{+} = \frac{2}{\kappa},\ \beta_{03}^{+} = -\frac{2(H_0+4)}{9\kappa^3},\ \beta_{05}^{+} = \frac{(H_0+4)(9H_0+4)}{675\kappa^5},\ \ldots,$$

$$\phi_0 \sim \phi_{02}^{+}\eta + \phi_{04}^{+}\eta^2 + \phi_{06}^{+}\eta^3 + \ldots \to 0,$$

$$\text{with } \phi_{02}^{+} = 1,\ \phi_{04}^{+} = -\frac{2(H_0+4)}{3\kappa^2},\ \phi_{06}^{+} = \frac{8(H_0+4)(3H_0+8)}{135\kappa^4},\ \ldots,$$

$$\text{as } \eta \to 0. \tag{5.13b}$$

5.2 The Zeroth-Order Approximation for the Inner Layer

A similarity formulation for this approximation exists. Based on (4.44) and (4.45), for this similarity formulation, the independent variables are

$$(x,\zeta),\ \text{with } \zeta = \zeta(x,r) = \frac{r}{h_{p0}^{*}(x)} = \left(\frac{dp_0^{*}}{dx}\right)^{1/3} r\ ; \tag{5.14}$$

and the dependent variables are

$$\phi_0(x,r) = g_0(\zeta);$$

$$m_0 = \frac{\partial\phi_0}{\partial r} = \left[\left(\frac{dp_0^{*}}{dx}\right)^{1/3}\right]\frac{dg_0}{d\zeta}\ ,$$

$$n_0 = -\frac{\partial\phi_0}{\partial x} = \left[\left(\frac{dp_0^{*}}{dx}\right)^{1/3}\right]\left[(1-2p_0^{*})^{-1}\left(\frac{dp_0^{*}}{dx}\right)^{2/3}\right]\left[\frac{1}{3}(H_0+4)\zeta\frac{dg_0}{d\zeta}\right], \tag{5.15a}$$

$$t_0(x,r) = \left(\frac{dp_0^{*}}{dx}\right)^{2/3}\omega_0(\zeta)\ . \tag{5.15b}$$

In terms of these new variables, with $q_0(x,r) = q_0^{+}(x) = p_0^{*}$, the zeroth-order inner-layer boundary-value problem becomes

$$\omega_0 = \zeta + \omega^{+} - \frac{d^2 g_0}{d\zeta^2}\ ; \tag{5.16}$$

$$\frac{dg_0}{d\zeta} \to 0,\ \omega_0 \to 0 \quad \text{as } \zeta \to 0, \tag{5.17a}$$

$$\frac{dg_0}{d\zeta} \sim b_{01}^{+}\zeta^{1/2} + b_{00}^{+} + a_{01}^{+}\zeta^{-1/2} + a_{02}^{+}\zeta^{-1} + \ldots \to \infty,$$

$$\text{with } b_{01}^{+} = \frac{2}{\kappa},\ b_{00}^{+} = \gamma^{+},\ a_{01}^{+} = -\frac{\omega^{+}}{\kappa},\ a_{02}^{+} = \frac{1}{2\kappa^2},\ \ldots,$$

$$\omega_0 \sim \omega_{02}^{+}\zeta + \omega_{00}^{+} + \vartheta_{01}^{+}\zeta^{-1/2} + \ldots \to \infty,$$

$$\text{with } \omega_{02}^{+} = 1,\ \omega_{00}^{+} = \omega^{+},\ \vartheta_{01}^{+} = -\frac{1}{\kappa},\ \ldots,\ \text{as } \zeta \to \infty. \tag{5.17b}$$

Here,

$$\omega_{00}^{+} = \omega^{+} = \left(\frac{d^2g_0}{d\zeta^2}\right)^{0} = \left[t_0^{+}\left(\frac{dp_0^{*}}{dx}\right)^{-2/3}\right] = \text{const.}, \tag{5.18a}$$

$$b_{00}^{+} = \gamma^{+} = \left(\frac{dg_0}{d\zeta}\right)^{+} = \left[m_0^{+}\left(\frac{dp_0}{dx}\right)^{-1/3}\right] = \text{const.} \tag{5.18b}$$

With respect to the similarity length-scale function, $\theta_0 = \bar{\lambda}_0 = \omega_0^{1/2}(d^2g_0/d\zeta^2)^{-1}$, the following asymptotic behaviors hold:

$$\theta_0 = \omega_0^{1/2}\left(\frac{d^2g_0}{d\zeta^2}\right)^{-1} \to \kappa\zeta(1 + \ldots) \to \infty \text{ as } \zeta \to \infty; \tag{5.19a}$$

$$\theta_0 = \omega_0^{1/2}\left(\frac{d^2g_0}{d\zeta^2}\right)^{-1} \sim \theta_{0s}^{0}\,\zeta^{s}(1 + \ldots) \to 0 \text{ as } \zeta \to 0,$$

$$\text{with } s,\ \theta_{0s}^{0},\ \ldots = \text{consts.} \tag{5.19b}$$

These behaviors suggest the following approximate closure:

$$\theta_0 = \omega_0^{1/2}\left(\frac{d^2g_0}{d\zeta^2}\right)^{-1} \approx \kappa\zeta E(\zeta;\ \ldots),$$

$$\text{with } E = [1 - \exp\{-\zeta(\omega^{+} + \zeta)^{1/2}/Z\}],\ Z = \text{const.} \tag{5.20}$$

For this closure, $\theta_0 \to \kappa\zeta(1 + \ldots)$ as $\zeta \to \infty$, and $\theta_0 \sim \theta^0_{0s}\zeta^s(1 + \ldots)$ as $\zeta \to 0$, with $s = 2$, $\theta^0_{0s} = \theta^{0}_{02} = (\kappa\omega^{+1/2}/Z)$. For this closure hypothesis, (5.16) takes the form

$$\kappa^2\zeta^2 E^2 \left(\frac{d^2 g_0}{d\zeta^2}\right)^2 + \left(\frac{d^2 g_0}{d\zeta^2}\right) - (\omega^+ + \zeta) = 0. \tag{5.21}$$

Thus,

$$\frac{d^2 g_0}{d\zeta^2} = \frac{2(\omega^+ + \zeta)}{\{1 + 4\kappa^2\zeta^2(\omega^+ + \zeta)E^2\}^{1/2} + 1}; \tag{5.22a}$$

$$\frac{dg_0}{d\zeta} = \int_0^\zeta \frac{2(\omega^+ + z)\, dz}{\{1 + 4\kappa^2 z^2(\omega^+ + z)E^2\}^{1/2} + 1}. \tag{5.22b}$$

For $\zeta \to 0$, the asymptotic behaviors of $(dg_0/d\zeta)$ and ω_0 are now determined to be

$$\frac{dg_0}{d\zeta} \sim \zeta \left\{(\omega^+ + \tfrac{1}{2}\zeta) - \left(\frac{\kappa^2\omega^{+3}}{5Z^2}\right)\zeta^4 \left[1 + \frac{5}{2\omega^+}\left(1 - \frac{\omega^{+3/2}}{3Z}\right)\zeta + \ldots\right]\right\}; \tag{5.23a}$$

$$\omega_0 \sim \left(\frac{\kappa^2\omega^{+3}}{Z^2}\right)\zeta^4 \left[1 + \frac{3}{\omega^+}\left(1 - \frac{\omega^{+3/2}}{3Z}\right)\zeta + \ldots\right]. \tag{5.23b}$$

For $\zeta \to \infty$, the asymptotic behaviors of $(dg_0/d\zeta)$ and ω_0 in this limit, as given in (5.17b), hold.

Note that, for the closure of (5.20),

$$\gamma^+ = \int_0^\infty \left[\frac{2(\omega^+ + \zeta)}{\{1 + 4\kappa^2\zeta^2(\omega^+ + \zeta)E^2\}^{1/2} + 1} - \frac{1}{\kappa\zeta^{1/2}}\right] d\zeta . \tag{5.24}$$

6. HIGHER-ORDER-APPROXIMATIONS SIMILARITY FORMULATIONS

6.1 Higher-Order Approximations for the Outer Layer

The equations of motion for $g_1(x,y)$, with $g_1 = u_1, v_1, p_1, \tau_{ij1}$, have been given previously [see (4.9)]. From consideration of the matching with the exterior layer,

$$u_1, p_1, \tau_{ij1} \to 0 \text{ as } y \to \infty. \tag{6.1}$$

Thus, $p_1 = p_1^* = \mathrm{fnc}(x) = 0$. From matching of the outer layer with the inner layer [see (4.48)],

$$u_1 \to m_0^+ = \gamma^+ \left(\frac{dp_0^*}{dx}\right)^{1/3}, \; \tau_1 \to 0 \text{ as } y \to 0. \tag{6.2}$$

Recall that γ^+ = const. and $(dp_0^*/dx) = \mathrm{fnc}(x;H_0)$ are introduced in the zeroth-order outer- and inner-layer similarity formulations.

For the first-order outer-layer similarity formulation, the independent variables are transformed from (x,y) to (x,η), where $\eta(x,y)$ is defined in (5.1). The first-order dependent variables are

$$\psi_1(x,y) = \psi_1(x,\eta) = \left[(1-2p_0^*)\left(\frac{dp_0^*}{dx}\right)^{-2/3}\right] f_1(\eta);$$

$$u_1 = \left[\left(\frac{dp_0^*}{dx}\right)^{1/3}\right] \frac{df_1}{d\eta},$$

$$v_1 = \left[\left(\frac{dp_0^*}{dx}\right)^{1/3}\right]\left[(H_0 + 2)\eta \frac{df_1}{d\eta} - \frac{2}{3}(H_0 + 1)f_1\right], \tag{6.3a}$$

$$\tau_1(x,y) = \tau_1(x,\eta) = \left[(1-2p_0^*)^{1/2}\left(\frac{dp_0^*}{dx}\right)^{1/3}\right] \phi_1(\eta). \tag{6.3b}$$

Introduction of (6.3) and (5.2) into (4.9), subject to (6.1) and (6.2), produces

$$\frac{d\phi_1}{d\eta} = -\frac{1}{3}(H_0 + 7)\frac{df_0}{d\eta}\frac{df_1}{d\eta} - (H_0 + 1)[f_0 \frac{d^2f_1}{d\eta^2} + \frac{2}{3}\frac{d^2f_0}{d\eta^2} f_1] ; \qquad (6.4)$$

$$\frac{df_1}{d\eta} \to 0,\ \phi_1 \to (K\Delta_\eta)\frac{d^2f_1}{d\eta^2} \to 0 \text{ as } \eta \to \infty , \qquad (6.5a)$$

$$\frac{df_1}{d\eta} \to \gamma^+,\ \phi_1 \to 2\kappa^2\eta^2 \frac{d^2f_0}{d\eta^2}\frac{d^2f_1}{d\eta^2} \to 0 \text{ as } \eta \to 0. \qquad (6.5b)$$

From (6.4) and (6.5), the inner-edge ($\eta \to 0$) behaviors of the velocity and stress functions are

$$\frac{df_1}{d\eta} \sim \beta^+_{10} + \beta^+_{12}\eta + \dots \to \beta^+_{10},$$

$$\text{with } \beta^+_{10} = b^+_{00} = \gamma^+,\ \beta^+_{12} = -\frac{4\gamma^+(H_0+4)}{9\kappa^2}, \dots , \qquad (6.6a)$$

$$\phi_1 \sim \phi^+_{13}\eta^{3/2} + \phi^+_{15}\eta^{5/2} + \dots \to 0,$$

$$\text{with } \phi^+_{13} = -\frac{8\gamma^+(H_0+4)}{9\kappa}, \dots . \qquad (6.6b)$$

The equations of motion for $g_2(x,y)$, with $g_2 = u_2, v_2, p_2, \tau_{ij2}$, are presented in (4.10). Since $p_2 \to p^*_2 = P^*_1 = \text{fnc}(x)$, and $\tau_{yy0} \to 0$ as $y \to \infty$, integration of (4.10c) yields

$$p_2 = p^*_2 + \tau_{yy0} . \qquad (6.7)$$

In turn, (4.10b) becomes

$$(u_0 \frac{\partial u_2}{\partial x} + v_0 \frac{\partial u_2}{\partial y} + \frac{\partial u_0}{\partial x} u_2 + \frac{\partial u_0}{\partial y} v_2) + \frac{dp^*_2}{dx} - \frac{\partial \tau_2}{\partial y}$$

$$= \frac{\partial \tau''_0}{\partial x} - (u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y}),\ \text{with } \tau''_0 = (\tau_{xx0} - \tau_{yy0}) \qquad (6.8)$$

Matching with the exterior layer [see (3.13)] produces

$$\frac{d(p_2^* + u_0^* u_2^*)}{dx} = 0: \quad p_2^* + u_0^* u_2^* = P_1^* + U_0^* U_1^* = \text{const.} = 0,$$

$$\text{with} \quad u_2^* = U_1^* \equiv W_2 \, [(1-2p_0^*)^{-1/2} \, (\frac{dp_0^*}{dx})^{2/3} \,],$$

$$p_2^* = P_1^* \equiv - W_2 \, (\frac{dp_0^*}{dx})^{2/3} \quad . \tag{6.9}$$

Similarly, matching with the inner layer [see (4.48) and (5.18)] produces

$$u_2 \to - \frac{1}{\kappa} t_0^+ \, (\frac{dp_0^*}{dx})^{-1/2} \, y^{-1/2} = - \frac{\omega^+}{\kappa} [(1-2p_0^*)^{-1/2} \, (\frac{dp_0^*}{dx})^{2/3} \,] \, \eta^{-1/2},$$

$$\tau_2 \to t_0^+ = \omega^+ \, (\frac{dp_0^*}{dx})^{2/3} \quad \text{as } y, \ \eta \to 0. \tag{6.10}$$

For the second-order outer-layer similarity formulation, the independent variables are transformed from (x,y) to (x,η), and the dependent variables are

$$\psi_2(x,y) = \psi_2(x,\eta) = [(1-2p_0^*)^{1/2} \, (\frac{dp_0^*}{dx})^{-1/3} \,] \, f_2(\eta) \ :$$

$$u_2 = [(1-2p_0^*)^{-1/2} \, (\frac{dp_0^*}{dx})^{2/3} \,] \, \frac{df_2}{d\eta},$$

$$v_2 = [(1-2p_0^*)^{-1/2} \, (\frac{dp_0^*}{dx})^{2/3} \,] \, [(H_0 + 2)\eta \, \frac{df_2}{d\eta} - \frac{1}{3} \, (H_0 + 1) \, f_2] \ , \tag{6.11a}$$

$$\tau_2(x,y) = \tau_2(x,\eta) = [(\frac{dp_0^*}{dx})^{2/3} \,] \, \phi_2(\eta) \ , \tag{6.11b}$$

$$\tau_0''(x,y) = \tau_0''(x,\eta) = [(\frac{dp_0^*}{dx})^{2/3} \,] \, \phi_0''(\eta) \ . \tag{6.11c}$$

Note that, with $p_2(x,y) = p_2(x,\eta) = [(dp_0^*/dx)^{2/3}]\pi_2(\eta) = -W_2(dp_0^*/dx)^{2/3}$,

$$\tau_{yy0}(x,y) = \tau_{yy0}(x,\eta) = [(dp_0^*/dx)^{2/3}][\pi_2(\eta) + W_2] = 0, \text{ and}$$

$$\tau_{xx0}(x,y) = \tau_{xx0}(x,\eta) = [(dp_0^*/dx)^{2/3}][\{\phi_0''(\eta) + \pi_2(\eta)\} + W_2] =$$

$$(dp_0^*/dx)^{2/3}\phi_0''(\eta).$$

The following boundary-value problem results:

$$\frac{d\phi_2}{d\eta} = \frac{2}{3}(H_0 + 4)\ [W_2 - \frac{df_0}{d\eta}\frac{df_2}{d\eta}] - (H_0 + 1)\ [f_0\frac{d^2f_2}{d\eta^2} + \frac{1}{3}\frac{d^2f_0}{d\eta^2}f_2]$$

$$- \frac{1}{3}(H_0 + 4)(\frac{df_1}{d\eta})^2 - \frac{2}{3}(H_0 + 1)\ f_1\frac{d^2f_1}{d\eta^2}$$

$$+ (H_0 + 2)\eta\frac{d\phi_0''}{d\eta} + \frac{2}{3}(H_0 + 4)\phi_0'' ; \tag{6.12}$$

$$\frac{df_2}{d\eta} \to W_2,\ \phi_2 \to (K\Delta_\eta)\frac{d^2f_2}{d\eta^2} \to 0 \text{ as } \eta \to \infty, \tag{6.13a}$$

$$\frac{df_2}{d\eta} \to -\frac{\omega^+}{\kappa}\eta^{-1/2},\ \phi_2 \to 2\kappa^2\eta^2\ [\frac{d^2f_0}{d\eta^2}\frac{d^2f_2}{d\eta^2} + \frac{1}{2}(\frac{d^2f_1}{d\eta^2})^2] \to \omega^+ \text{ as } \eta \to 0. \tag{6.13b}$$

The inner-edge ($\eta \to 0$) behaviors for the velocity and stress functions are

$$\frac{df_2}{d\eta} \sim \alpha_{21}^+\eta^{-1/2} + \beta_{21}^+\eta^{1/2} + \dots ,$$

$$\text{with } \alpha_{21}^+ = a_{01}^+ = -\frac{\omega^+}{\kappa},\ \beta_{21}^+ = \frac{2}{3\kappa}(W_2 + \frac{5\omega^+}{2\kappa^2} - \frac{\gamma^{+2}}{2})(H_0 + 4), \dots ,$$

$$\phi_2 \sim \phi_{20}^+ + \phi_{22}^+\eta + \dots ,$$

$$\text{with } \phi_{20}^+ = \omega_{00}^+ = \omega^+,\ \phi_{22}^+ = \frac{2}{3}(W_2 + \frac{2\omega^+}{\kappa^2} - \frac{\gamma^{+2}}{2})(H_0 + 4), \dots . \tag{6.14}$$

In determining these behaviors, it has been taken that $\phi_0'' \sim \phi_{02}''^{+}\eta + \ldots$ as $\eta \to 0$.

The equations of motion for $g_3(x,y)$, with $g_3 = u_3, v_3, p_3, \tau_{ij3}$, are presented in (4.11). From consideration of the matching with the exterior layer,

$$u_3,\ p_3,\ \tau_{ij3} \to 0 \text{ as } y \to \infty. \tag{6.15}$$

Based on (6.15), integration of (4.11c) yields

$$p_3 = \tau_{yy1} \ . \tag{6.16}$$

In turn, (4.11b) becomes

$$\begin{aligned}(u_0 \frac{\partial u_3}{\partial x} + v_0 \frac{\partial u_3}{\partial y} + \frac{\partial u_0}{\partial x} u_3 + \frac{\partial u_0}{\partial y} v_3) - \frac{\partial \tau_3}{\partial y} \\ = \frac{\partial^2 u_0}{\partial y^2} + \frac{\partial \tau_1''}{\partial x} - (u_1 \frac{\partial u_2}{\partial x} + v_1 \frac{\partial u_2}{\partial y} + \frac{\partial u_1}{\partial x} u_2 + \frac{\partial u_1}{\partial y} v_2),\end{aligned}$$

$$\text{with } \tau_1'' = (\tau_{xx1} - \tau_{yy1}). \tag{6.17}$$

Matching with the inner layer produces

$$u_3 \to \frac{1}{2\kappa^2} y^{-1} = \frac{1}{2\kappa^2}[(1-2p_0^*)^{-1} \frac{dp_0^*}{dx}]\eta^{-1},$$

$$\tau_3 \to -\frac{1}{\kappa}(\frac{dp_0^*}{dx})^{1/2} y^{-1/2} = -\frac{1}{\kappa}[(1-2p_0^*)^{-1/2} \frac{dp_0^*}{dx}]\eta^{-1/2} \text{ as } y,\ \eta \to 0. \tag{6.18}$$

For the third-order outer-layer similarity formulation, with x, η as the independent variables, the dependent variables are

$$\psi_3(x,y) = \psi_3(x,\eta) = f_3(\eta):$$

$$u_3 = [(1-2p_0^*)^{-1} \frac{dp_0^*}{dx}] \frac{df_3}{d\eta} ,$$

$$v_3 = [(1-2p_0^*)^{-1} \frac{dp_0^*}{dx}][(H_0 + 2)\eta \frac{df_3}{d\eta}], \tag{6.19a}$$

$$\tau_3(x,y) = \tau_3(x,\eta) = [(1-2p_0^*)^{-1/2} \frac{dp_0^*}{dx}] \phi_3(\eta) , \tag{6.19b}$$

$$\tau_1''(x,y) = \tau_1''(x,\eta) = [(1-2p_0^*)^{-1/2} \frac{dp_0^*}{dx}] \phi_1''(\eta) . \tag{6.19c}$$

With $p_3(x,y) = p_3(x,\eta) = [(1-2p_0^*)^{-1/2}(dp_0^*/dx)]\pi_3(\eta) = 0,$

$$\tau_{yy1}(x,y) = \tau_{yy1}(x,\eta) = [(1-2p_0^*)^{-1/2}(dp_0^*/dx)]\pi_3(\eta) = 0, \text{ and}$$

$$\tau_{xx1}(x,y) = \tau_{xx1}(x,\eta) = [(1-2p_0^*)^{-1/2}(dp_0^*/dx)] [\phi_1''(\eta) + \pi_3(\eta)]$$

$$= [(1-2p_0^*)^{-1/2}(dp_0^*/dx)]\phi_1''(\eta).$$

The following boundary-value problem results:

$$\begin{aligned}\frac{d\phi_3}{d\eta} = &- \frac{d^3f_0}{d\eta^3} \\ &- (H_0 + 3) \frac{df_0}{d\eta} \frac{df_3}{d\eta} - (H_0 + 1)f_0 \frac{d^2f_3}{d\eta^2} \\ &- (H_0 + 3) \frac{df_1}{d\eta} \frac{df_2}{d\eta} - \frac{2}{3} (H_0 + 1)[f_1 \frac{d^2f_2}{d\eta^2} + \frac{1}{2} \frac{d^2f_1}{d\eta^2} f_2] \\ &+ (H_0 + 2)\eta \frac{d\phi_1''}{d\eta} + (H_0 + 3)\phi_1'' ,\end{aligned} \tag{6.20}$$

$$\frac{df_3}{d\eta} \to 0, \ \phi_3 \to (K\Delta_\eta) \frac{d^2f_3}{d\eta^2} \text{ as } \eta \to \infty , \tag{6.21a}$$

$$\frac{df_3}{d\eta} \to \frac{1}{2\kappa^2} \eta^{-1} ,$$

$$\phi_3 \to 2\kappa^2\eta^2 \left[\frac{d^2f_0}{d\eta^2}\frac{d^2f_3}{d\eta^2} + \frac{d^2f_1}{d\eta^2}\frac{d^2f_2}{d\eta^2} - \left(\frac{d^2f_1}{d\eta^2}\right)^{-1}\left(\frac{d^2f_1}{d\eta^2}\right)^3\right]$$

$$\to -\frac{1}{\kappa}\eta^{-1/2} \text{ as } \eta \to 0. \tag{6.21b}$$

The inner-edge ($\eta \to 0$) behaviors for the velocity and stress functions are

$$\frac{df_3}{d\eta} \sim \alpha_{32}^+\eta^{-1} + \dots, \text{ with } \alpha_{32}^+ = a_{02}^+ = \frac{1}{2\kappa^2}, \dots,$$

$$\phi_3 \sim \lambda_{31}^+\eta^{-1/2} + \dots, \text{ with } \lambda_{31}^+ = \vartheta_{01}^+ = -\frac{1}{\kappa}, \dots . \tag{6.22}$$

Here, it is suggested that $\phi_1'' \sim \phi_{13}''^{+}\eta^{3/2} + \dots$ as $\eta \to 0$.

To summarize the results obtained for the outer-layer similarity formulations, it is convenient to introduce the quantity

$$\Lambda_* = \delta \left[(1-2p_0^*)^{-1}\left(\frac{dp_0^*}{dx}\right)^{2/3}\right]$$

$$= \delta \left[(H_0+2)\, u_{00}^*(x-x_0)^{(H_0+1)/H_0+2)}\right]^{-2/3}, \tag{6.23}$$

such that $\Lambda_* \sim O(\delta) << 1$ for x fixed. In turn, the outer-layer streamwise-velocity and Reynolds-stress functions take the following forms:

$$\frac{u}{u_0^*} \cong \frac{df_0}{d\eta} + \Lambda_*^{1/2}\frac{df_1}{d\eta} + \Lambda_*\frac{df_2}{d\eta} + \Lambda_*^{3/2}\frac{df_3}{d\eta} + \dots ; \tag{6.24a}$$

$$\frac{\tau}{u_0^{*2}} \cong \phi_0 + \Lambda_*^{1/2}\phi_1 + \Lambda_*\phi_2 + \Lambda_*^{3/2}\phi_3 + \dots . \tag{6.24b}$$

Recall that $u_0^* = (1-2p_0^*)^{1/2} = u_{00}^*(x - x_0)^{-1/(H_0+2)}$. The inner-edge ($\eta \to 0$) behaviors for these velocity and stress functions are

$$\frac{u_*}{u_0} \sim [\beta_{01}^+\eta^{1/2} + \beta_{03}^+\eta^{3/2} + \beta_{05}^+\eta^{5/2} + \ldots] + \Lambda_*^{1/2}\,[\beta_{10}^+ + \beta_{12}^+\eta + \ldots]$$

$$+ \Lambda_*\,[\alpha_{21}^+\eta^{-1/2} + \beta_{21}^+\eta^{1/2} + \ldots] + \Lambda_*^{3/2}\,[\alpha_{32}^+\eta^{-1} + \ldots] + \ldots\ ; \quad (6.25a)$$

$$\frac{\tau}{u_0^{*2}} \sim [\phi_{02}^+\eta + \phi_{04}^+\eta^2 + \phi_{06}^+\eta^3 + \ldots] + \Lambda_*^{1/2}[\phi_{13}^+\eta^{3/2} + \ldots]$$

$$+ \Lambda_*\,[\phi_{20}^+ + \phi_{22}^+\eta + \ldots] + \Lambda_*^{3/2}[\lambda_{31}^+\eta^{-1/2} + \ldots] + \ldots\ . \quad (6.25b)$$

With the introduction of Λ_*, the outer-layer coordinate, η, and the inner-layer coordinate, ζ, are related by

$$\eta = \Lambda_*\zeta \quad \text{and/or} \quad \zeta = \Lambda_*^{-1}\eta\ . \quad (6.26)$$

For consideration of the corresponding inner-layer velocity and stress functions, it is convenient to introduce

$$u_{p0}^* = \delta^{1/2}\left(\frac{dp_0^*}{dx}\right)^{1/3}$$

$$= \delta^{1/2}u_{00}^*\ [(H_0 + 2)u_{00}^*(x - x_0)^{(H_0+4)/(H_0+2)}]^{-1/3}. \quad (6.27)$$

From (6.25)-(6.27), it is determined that the outer-edge ($\zeta \to \infty$) behaviors of the inner-layer velocity and stress functions are

$$\frac{u}{u_{p0}^*} \sim [\beta_{01}^+\zeta^{1/2} + \beta_{10}^+ + \alpha_{21}^+\zeta^{-1/2} + \alpha_{32}^+\zeta^{-1} + \ldots]$$

$$+ \Lambda_*\,[\beta_{03}^+\zeta^{3/2} + \beta_{12}^+\zeta + \beta_{21}^+\zeta^{1/2} + \ldots]$$

$$+ \Lambda_*^2\,[\beta_{05}^+\zeta^{5/2} + \ldots] + \ldots\ ; \quad (6.28a)$$

$$\frac{\tau}{u_{p0}^{*2}} \sim [\phi_{02}^{+}\zeta + \phi_{20}^{+} + \lambda_{31}^{+}\zeta^{-1/2} + \ldots]$$

$$+ \Lambda_{*}[\phi_{04}^{+}\zeta^{2} + \phi_{13}^{+}\zeta^{3/2} + \phi_{22}^{+}\zeta + \ldots]$$

$$+ \Lambda_{*}^{2}\,[\phi_{06}^{+}\zeta^{3} + \phi_{15}^{+}\,\zeta^{5/2} + \ldots] + \ldots . \qquad (6.28b)$$

6.2 Higher-Order Approximations for the Inner Layer

The equations of motion for the first-, second-, and third-order approximations for the inner layer are presented in (4.32) to (4.34).

For the first-order approximation, the first integrals of the momentum equations, when the surface boundary conditions are taken into account, are,

$$q_1 = q_1^0 = \mathrm{fnc}(x), \qquad (6.29a)$$

$$t_1 + \frac{\partial m_1}{\partial r} - \frac{dq_1^0}{dx}\, r = \left(\frac{\partial m_1}{\partial r}\right)^0 = \mathrm{fnc}(x). \qquad (6.29b)$$

From matching with the outer layer [see (6.28)], it follows that

$$m_1 \to 0,\ t_1 \to 0,\ q_1 = q_1^0 \to 0 \text{ as } r \to \infty. \qquad (6.30)$$

Thus, the first-order inner-layer solutions are

$$m_1(x,r) = 0,\ t_1(x,r) = 0,\ q_1(x,r) = q_1^0(x) = 0. \qquad (6.31a\text{-}c)$$

For the second-order approximation, the first integrals of the momentum equations are

$$q_2 = q_2^0 = \mathrm{fnc}(x), \qquad (6.32a)$$

$$t_2 + \frac{\partial m_2}{\partial r} - \frac{dq_2^0}{dx}\, r - \int_0^r \left(m_0 \frac{\partial m_0}{\partial x} + n_0 \frac{\partial m_0}{\partial r_1}\right) dr_1 = \left(\frac{\partial m_2}{\partial r}\right)^0 = \mathrm{fnc}(x). \qquad (6.32b)$$

Matching with the outer layer produces

$$m_2 \to \frac{2}{9\kappa^3}\left(\frac{dp_0^*}{dx}\right)^{-1/2}\frac{d^2p_0^*}{dx^2}\,r^{3/2} = -\frac{2(H_0+4)}{9\kappa^3}\left[(1-2p_0^*)^{-1}\frac{dp_0^*}{dx}\right]\zeta^{3/2},$$

$$t_2 \to \frac{2}{3\kappa^2}\frac{d^2p_0^*}{dx^2}\,r^2 = -\frac{2(H_0+4)}{3\kappa^2}\left[(1-2p_0^*)^{-1}\left(\frac{dp_0^*}{dx}\right)^{4/3}\right]\zeta^2,$$

$$q_2 = q_2^0 \to -W_2\left(\frac{dp_0^*}{dx}\right)^{2/3} \quad \text{as } r,\ \zeta \to \infty . \tag{6.33}$$

Here, it is taken that

$$\phi_2(x,r) = \phi_2(x,\zeta) = \left[(1-2p_0^*)^{-1}\left(\frac{dp_0^*}{dx}\right)^{2/3}\right] g_2(\zeta):$$

$$m_2 = \left[(1-2p_0^*)^{-1}\frac{dp_0^*}{dx}\right]\frac{dg_2}{d\zeta} , \tag{6.34a}$$

$$t_2(x,r) = t_2(x,\zeta) = \left[(1-2p_0^*)^{-1}\left(\frac{dp_0^*}{dx}\right)^{4/3}\right]\omega_2(\zeta) , \tag{6.34b}$$

$$q_2(x,r) = q_2^+(x) = -W_2\left(\frac{dp_0^*}{dx}\right)^{2/3} . \tag{6.34c}$$

With $\omega_2^+ = [(1-2p_0^*)(dp_0^*/dx)^{-4/3}](\partial m_2/\partial r)^0$ = const., the second-order inner-layer boundary-value problem is

$$\omega_2 = \frac{2}{3}(H_0+4)\left[W_2\zeta - \frac{1}{2}\int_0^\zeta \left(\frac{dg_0}{d\zeta_1}\right)^2 d\zeta_1\right] + \omega_2^+ - \frac{d^2g_2}{d\zeta^2} ; \tag{6.35}$$

$$\frac{dg_2}{d\zeta} \to 0,\ \omega_2 \to 0 \text{ as } \zeta \to 0, \tag{6.36a}$$

$$\frac{dg_2}{d\zeta} \to -\frac{2(H_0+4)}{9\kappa^3}\zeta^{3/2},\ \omega_2 \to -\frac{2(H_0+4)}{3\kappa^2}\zeta^2 \text{ as } \zeta \to \infty . \tag{6.36b}$$

The outer-edge ($\zeta \to \infty$) behaviors for the velocity and stress functions are

$$\frac{dg_2}{d\zeta} \sim b_{23}^{+}\zeta^{3/2} + b_{22}^{+}\zeta + b_{21}^{+}\zeta^{1/2} + \ldots\ldots ,$$

$$\text{with } b_{23}^{+} = \beta_{03}^{+} = -\frac{2(H_0+4)}{9\kappa^3},\ b_{22}^{+} = \beta_{12}^{+} = -\frac{4\gamma^{+}(H_0+4)}{9\kappa^2},$$

$$b_{21}^{+} = \beta_{21}^{+} = \frac{2}{3\kappa}(W_2 + \frac{5\omega^{+}}{2\kappa^2} - \frac{\gamma^{+2}}{2})(H_0+4),\ \ldots ,$$

$$\omega_2 \sim \omega_{24}^{+}\zeta^2 + \omega_{23}^{+}\zeta^{3/2} + \omega_{22}^{+}\zeta + \ldots ,$$

$$\text{with } \omega_{24}^{+} = \phi_{04}^{+} = -\frac{2(H_0+4)}{3\kappa^2},\ \omega_{23}^{+} = \phi_{13}^{+} = -\frac{8\gamma^{+}(H_0+4)}{9\kappa},$$

$$\omega_{22}^{+} = \phi_{22}^{+} = \frac{2}{3}(W_2 + \frac{2\omega^{+}}{\kappa^2} - \frac{\gamma^{+2}}{2})(H_0 + 4),\ \ldots . \tag{6.37}$$

With m_1, $n_1 = 0$, the first integrals of the momentum equations for the third-order approximation are

$$q_3 = q_3^0 = fnc(x), \tag{6.38a}$$

$$t_3 + \frac{\partial m_3}{\partial r} - \frac{dq_3^0}{dx} r = (\frac{\partial m_3}{\partial r})^0 = fnc(x)\ . \tag{6.38b}$$

From matching with the outer layer, it is seen that

$$m_3 \to 0,\ t_3 \to 0,\ q_3 = q_3^0 \to 0 \text{ as } r \to \infty. \tag{6.39}$$

Thus, the third-order inner-layer solutions are

$$m_3(x,r) = 0,\ t_3(x,r) = 0,\ q_3(x,r) = q_3^0(x) = 0. \tag{6.40a-c}$$

In summary, then, the results for the streamwise-velocity and Reynolds-stress functions, from the inner-layer similarity formulations, can be expressed as

$$\frac{u}{u_{p0}^{*}} \cong \frac{dg_0}{d\zeta} + \Lambda_* \frac{dg_2}{d\zeta} + \dots ; \tag{6.40a}$$

$$\frac{\tau}{u_{p0}^{*2}} \cong \omega_0 + \Lambda_* \omega_2 + \dots . \tag{6.40b}$$

The outer-edge ($\zeta \to \infty$) behaviors for these velocity and stress functions are

$$\frac{u}{u_{p0}^{*}} \sim [b_{01}^{+}\zeta^{1/2} + b_{00}^{+} + a_{01}^{+}\zeta^{-1/2} + a_{02}^{+}\zeta^{-1} + \dots]$$

$$+ \Lambda_*[b_{23}^{+}\zeta^{3/2} + b_{22}^{+}\zeta + b_{21}^{+}\zeta^{1/2} + \dots] + \dots ; \tag{6.41a}$$

$$\frac{\tau}{u_{p0}^{*2}} \sim [\omega_{02}^{+}\zeta + \omega_{00}^{+} + \vartheta_{01}^{+}\zeta^{-1/2} + \dots]$$

$$+ \Lambda_*[\omega_{24}^{+}\zeta^{2} + \omega_{23}^{+}\zeta^{3/2} + \omega_{22}^{+}\zeta + \dots] + \dots . \tag{6.41b}$$

In (6.25) and (6.41),

$$\beta_{01}^{+} = b_{01}^{+} = \frac{2}{\kappa},\ \beta_{10}^{+} = b_{00}^{+} = \gamma^{+},\ \alpha_{21}^{+} = a_{01}^{+} = -\frac{\omega^{+}}{\kappa},\quad \alpha_{32}^{+} = a_{02}^{+} = \frac{1}{2\kappa^{2}}, \dots ,$$

$$\beta_{03}^{+} = b_{23}^{+} = -\frac{2(H_0+4)}{9\kappa^{3}},\ \beta_{12}^{+} = b_{22}^{+} = -\frac{4\gamma^{+}(H_0+4)}{9\kappa^{2}},$$

$$\beta_{21}^{+} = b_{21}^{+} = \frac{2}{3\kappa}(W_2 + \frac{5\omega^{+}}{2\kappa^{2}} - \frac{\gamma^{+2}}{2})(H_0 + 4), \dots , \dots ; \tag{6.42a}$$

$$\phi_{02}^{+} = \omega_{02}^{+} = 1,\ \phi_{20}^{+} = \omega_{00}^{+} = \omega^{+},\ \lambda_{31}^{+} = \vartheta_{01}^{+} = -\frac{1}{\kappa}, \dots ,$$

$$\phi_{04}^{+} = \omega_{24}^{+} = -\frac{2(H_0+4)}{3\kappa^{2}},\ \phi_{13}^{+} = \omega_{23}^{+} = -\frac{8\gamma^{+}(H_0+4)}{9\kappa},$$

$$\phi_{22}^{+} = \omega_{22}^{+} = \frac{2}{3}(W_2 + \frac{2\omega^{+}}{\kappa^{2}} - \frac{\gamma^{+2}}{2})(H_0 + 4), \dots , \dots . \tag{6.42b}$$

7. SOLUTIONS FOR THE DISTINGUISHED INTERMEDIATE LAYER

In Section 6, for the streamwise-velocity and Reynolds-stress functions, the inner-edge behaviors for the outer-layer solutions and the outer-edge behaviors for the inner-layer solutions are given. An examination of (6.25) and (6.41) suggests that a distinguished intermediate layer exists. For the distinguished similarity intermediate layer, the independent-variable transformation is $(x,y) \to (x,\chi)$, where

$$\chi = (\eta\zeta)^{1/2} = \Lambda_*^{-1/2}\eta = \Lambda_*^{1/2}\zeta$$

$$= [(1-2p_0^*)(\frac{dp_0^*}{dx})^{-4/3}]^{-1/2} (y/\delta^{1/2})$$

$$= [(1-2p_0^*)(\frac{dp_0^*}{dx})^{-4/3}]^{-1/2} (\delta^{1/2}r). \qquad (7.1)$$

The appropriate speed function for this intermediate layer is

$$u_{10}^* = (u_0^* u_{p0}^*)^{1/2} = \Lambda_*^{1/4}\, u_0^* = \Lambda_*^{-1/4}\, u_{p0}^*$$

$$= \delta^{1/4}[(1-2p_0^*)(\frac{dp_0^*}{dx})^{2/3}]^{1/4}$$

$$= \delta^{1/4}\, u_{00}^*\, [(H_0 + 2)\, u_{00}^*(x - x_0)^{(H_0+7)/(H_0+2)} {}^{-1/6}]. \qquad (7.2)$$

Introduction of $\chi = \Lambda_*^{-1/2}\,\eta$ into

$$(u/u_0^*) = \Lambda_*^{1/4}\,(u/u_{10}^*),\quad (\tau/u_0^{*2}) = \Lambda_*^{1/2}(\tau/u_{10}^{*2}),$$

and/or $\chi = \Lambda_*^{1/2}\zeta$ into

$$(u/u_{p0}^*) = \Lambda_*^{-1/4}\,(u/u_{10}^*),\quad (\tau/u_{p0}^{*2}) = \Lambda_*^{-1/2}\,(\tau/u_{10}^{*2}),$$

as given in (6.25) and/or (6.41), produces

$$\frac{u_*}{u_{10}} \cong [\beta^+_{01}\chi^{1/2}] + \Lambda_*^{1/4}[\beta^+_{10}]$$

$$+ \Lambda_*^{1/2}[\beta^+_{03}\chi^{3/2} + \alpha^+_{21}\chi^{-1/2}] + \Lambda_*^{3/4}[\beta^+_{12}\chi + \alpha^+_{32}\chi^{-1}] + \ldots ; \qquad (7.3a)$$

$$\frac{\tau}{u_{10}^{*2}} \cong [\phi^+_{02}\chi] + \Lambda_*^{1/2}[\phi^+_{04}\chi^2 + \phi^+_{20}] + \Lambda_*^{3/4}[\phi^+_{13}\chi^{3/2} + \lambda^+_{31}\chi^{-1/2}] + \ldots . \qquad (7.3b)$$

The results of Kader & Yaglom (1978) can be compared with those of the present paper. The notation of Kader & Yaglom in terms of the present notation is given by

$$(\alpha)_{KY} = \left(\frac{1}{\rho}\frac{dP}{dx}\right)_{KY} \to \left(\frac{\tilde{u}_\infty^2}{\tilde{c}}\right)\frac{dp_0^*}{dx},$$

$$(\delta)_{KY} \to \tilde{c}\delta h_0^* = \tilde{c}\delta\ [(1-2p_0^*)(\frac{dp_0^*}{dx})^{-1}] ,$$

$$(u_*)_{KY} = \left(\frac{\tau_w}{\rho}\right)^{1/2}_{KY} \to \tilde{u}_\infty\delta\omega^{+1/2}\left(\frac{dp_0^*}{dx}\right)^{1/3} ;$$

$$\left(\frac{u_*^2}{\alpha\delta}\right)_{KY} \to \delta\omega^+[(1-2p_0^*)^{-1}\ \left(\frac{dp_0^*}{dx}\right)^{2/3}] = \omega^+\Lambda_* \to 0;$$

$$(y)_{KY} \to \tilde{c}\ y;\ (\alpha y)^{1/2}_{KY} \to \tilde{u}_\infty\ \left(\frac{dp_0^*}{dx}\ y\right)^{1/2} = \tilde{u}_\infty\ u^*_{10}\chi^{1/2},$$

$$(U)_{KY} \to \tilde{u}_\infty\ u. \qquad (7.4)$$

In turn, the Kader-Yaglom representation for the streamwise velocity may be expressed as

$$[U = \mathcal{K}(\alpha y)^{1/2} + \mathcal{K}_1]_{KY}$$

$$\rightarrow \frac{u}{u^*_{i0}} \approx [4.5\ \{1 + 10(\omega^+\Lambda_*)\}^{1/2}\chi^{1/2}]$$

$$+ \Lambda_*^{1/4}[(\delta\omega^+)^{1/2}\{2.44 \log \Gamma - 15\ \Gamma^{-1/2} - 6\Gamma^{-1}\}],$$

$$\text{where } \Gamma = \frac{6(\delta^{-1/3}\omega^+)^{3/2}}{5\{1 + 10(\omega^+\Lambda_*)\}} . \tag{7.5}$$

The results of Afzal (1983) also can be compared to the present ones. The notation of Afzal in terms of the present notation is

$$(U_p)_A = (\frac{\nu p_x}{\rho})_A^{1/3} \rightarrow \tilde{u}_\infty \delta^{7/6}(\frac{dp^*_0}{dx})^{1/3} = \tilde{u}_\infty \delta^{2/3} u^*_{p0}$$

$$(\delta)_A \rightarrow \tilde{c}\delta h^*_0 = \tilde{c}\delta\ [(1-2p^*_0)\ (\frac{dp^*_0}{dx})^{-1}]\ :$$

$$(R_p)_A = (\frac{U_p\delta}{\nu})_A \rightarrow \delta^{-4/3}\ [(1-2p^*_0)^{-1}(\frac{dp^*_0}{dx})^{2/3}]^{-1} = \delta^{-1/3}\Lambda_*^{-1} \rightarrow \infty,$$

$$(\Lambda)_A = (\frac{\tau_w}{p_x\delta})_A \rightarrow \delta\omega^+[(1-2p^*_0)^{-1}\ (\frac{dp^*_0}{dx})^{2/3}] = \omega^+\Lambda_* \rightarrow 0,$$

$$(\Lambda R_p)_A \rightarrow \delta^{-1/3}\omega^+ \rightarrow \infty;$$

$$(y)_A \rightarrow \tilde{c}y:\ (\frac{U_p y}{\nu})_A^{1/2} \rightarrow \delta^{-7/6}\ (\frac{dp^*_0}{dx})^{1/6} y^{1/2} = \delta^{-2/3}\Lambda_*^{-1/4}\chi^{1/2},$$

$$(u)_A \rightarrow \tilde{u}_\infty u:\ (\frac{u}{U_p})_A \rightarrow \delta^{-2/3}\Lambda_*^{-1/4}\ \frac{u}{u^*_{i0}} . \tag{7.6}$$ †

† In Afzal's analysis, it is taken that $(\Lambda)_A \rightarrow 0$, $(R_p)_A \rightarrow \infty$, and $(\Lambda R_p)_A \sim O(1)$. Note that $(u_*^2/\alpha\delta)_{KY} = (\Lambda)_A = \omega^+\Lambda_* \rightarrow 0$.

Afzal's inner-layer representation for the streamwise velocity and its expression in the present intermediate-layer notation are

$$\left[\frac{u}{U_p} \approx 3.5\ (1 + 5.4\Lambda)\ \left(\frac{U_p y}{\nu}\right)^{1/2} + 2.5\ (\Lambda R_p)\right]_A$$

$$\rightarrow \frac{u}{u_{*10}} \approx [3.5\ \{1 + 5.4\ (\omega^+\Lambda_*)\}\chi^{1/2}] + \Lambda_*^{1/4}[2.5(\delta^{1/3}\omega^+)] \quad . \qquad (7.7)$$

The representation of Stratford (1959b) for the streamwise velocity in the intermediate layer is

$$\left[u \approx \frac{2}{\kappa}\left(\frac{1}{\rho}\frac{dp}{dx}\right)^{1/2} y^{1/2} + C\left(\frac{\nu}{\rho}\frac{dp}{dx}\right)^{1/3}\right]_S, \text{ with } C = \text{const.}$$

$$\rightarrow \frac{u}{u_{*10}} \approx \left[\frac{2}{\kappa}\ \chi^{1/2}\right] + \Lambda_*^{1/4}[C\ \delta^{2/3}] \quad . \qquad (7.8)$$

Thus the intermediate-layer streamwise-velocity representations of Kader & Yaglom, Afzal, and Stratford, i.e., (7.5), (7.7), and (7.8), respectively, are compared to the present one, (7.3a). The comparison indicates that the three-layer theory provides an improved characterization of the overlap domain over that provided by the (classical) two-layer theory.

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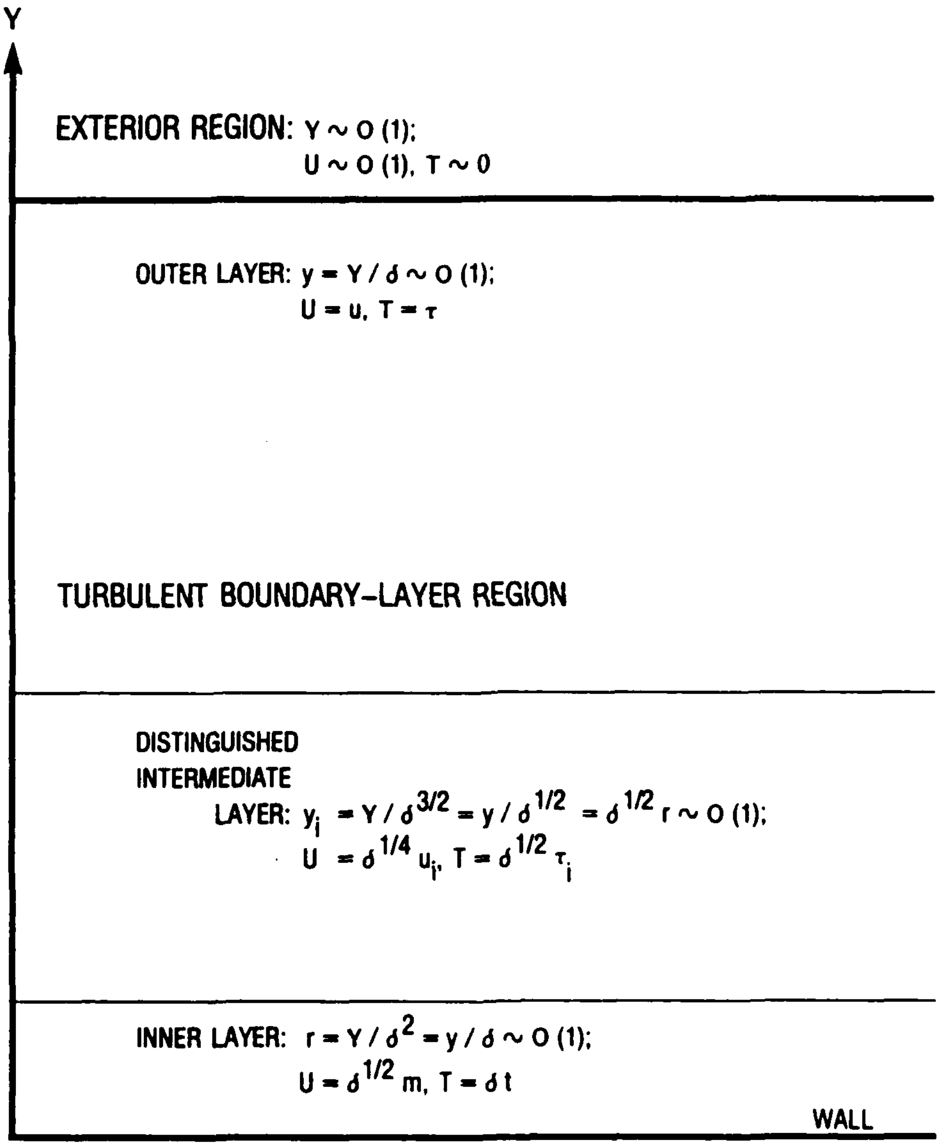

Figure 1. Schematic diagram of the asymptotic structure of the boundary layer.

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